THE NON-ORTHOGONAL CAYLEY-DICKSON CONSTRUCTION AND THE OCTONIONIC STRUCTURE OF THE E_8-LATTICE

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Abstract. Using a conic (= degree-2) algebra B over an arbitrary commutative ring, a scalar μ and a linear form s on B as input, the non-orthogonal Cayley-Dickson construction produces a conic algebra $C := \mathrm{Cay}(B; \mu, s)$ and collapses to the standard (orthogonal) Cayley-Dickson construction for $s = 0$. Conditions on B, μ, s that are necessary and sufficient for C to satisfy various algebraic properties (like associativity or alternativity) are derived. Sufficient conditions guaranteeing non-singularity of C even if B is singular are also given. As an application we show how the algebras of Hurwitz quaternions and of Dickson or Coxeter octonions over the rational integers can be obtained from the non-orthogonal Cayley-Dickson construction.

Contents

1. Introduction

The arithmetic background of the present paper is dominated by the E_8-lattice, the unique indecomposable unimodular positive definite quadratic lattice of rank 8 over the integers that has recently come to the fore again when Viazovska [19] proved that the densest sphere packing of eight-dimensional euclidean space is the E_8-lattice sphere packing, having density $\pi^4/384$. Our aim here is much more modest, focusing instead on a discovery attributed to Coxeter [2] (see also Pumplün [17]), but originally due to Dickson [3], to the effect that the E_8-lattice carries the structure of an octonion algebra over the integers whose generic fiber is the unique octonion division algebra over the rationals.

Working over an arbitrary commutative associative ring of scalars, our principal objective in this paper will be to describe an elementary, purely algebraic formalism, called the *non-orthogonal Cayley-Dickson construction*, that, among other things, provides an intrinisc approach to the Dickson and Coxeter octonions once the appropriate specifications have been made, see §9 below for details. This formalism, generalizing the classical Cayley-Dickson construction, one of the most versatile tools in all of non-associative algebra, has been investigated before by Garibaldi-Petersson [5, §4] under the more restrictive condition of a base field having characteristic 2.

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The classical Cayley-Dickson construction starts from what we call a conic algebra B (see 2.1 for the precise definition) and a scalar μ in the base ring as input to produce a new conic algebra as output, which we denote by $C := \mathrm{Cay}(B, \mu)$. One of the many interesting features of this construction is that, by passing from B to C we are bound to lose a considerable amount of algebraic information, but we will do so in a controlled manner. The usefulness of its non-orthogonal counterpart hinges on the question of whether the same amount of control can be guaranteed also under these more general circumstances. The bulk of the present work is devoted to answering this question in the affirmative. Unfortunately, we are able to achieve this objective only by an excessive amount of horrendous computations.

More specifically, the input of the non-orthogonal Cayley-Dickson construction, beside the data B, μ as above, consists of a linear form s acting on B, while the output is again a conic algebra, denoted by $C := \mathrm{Cay}(B; \mu, s)$. The main task we address ourselves to will then be to find conditions in terms of B, μ, s that are necessary and sufficient for the algebra C to be respectively commutative, associative or alternative. While the restrictions on B and μ are to be expected from the orthogonal case, it is the ones on s that make our investigation delicate and cumbersome. We refer to §§4–7 for details.

Throughout this paper, we fix an arbitrary commutative ring denoted by k. All k-algebras are assumed to be non-associative; their module structure is arbitrary

2. Conic algebras

In this section, we define the notion of a conic algebra and recall some of its most useful properties. Our main reference is [16].

2.1. **The concept of a conic algebra.** Adopting the terminology of Loos [10], we define a *conic algebra* over k, more commonly known under the name *algebra of degree* 2 (McCrimmon [12]) or *quadratic algebra* (Osborn [15]), as a unital k-algebra C together with a quadratic form $n_C\colon C \to k$, by abuse of language called the *norm* of C, such that $n_C(1_C) = 1$ and

$$x^2 - t_C(x)x + n_C(x)1_C = 0$$

for all $x \in C$. Here $t_C\colon C \to k$ is the *trace* of C defined as the linear form $x \mapsto n_C(1_C, x)$, where

$$(x, y) \mapsto n_C(x, y) := n_C(x + y) - n_C(x) - n_C(y)$$

stands for the bilinearization of n_C. We then define the *conjugation* of C as the map

$$\iota_C\colon C \longrightarrow C, \quad x \longmapsto \bar{x} := t_C(x)1_C - x,$$

which is linear of period 2 but will fail in general to be an (algebra) involution.

2.2. **Basic identities.** The following identities by [16, 18.5] hold in arbitrary conic algebras .

(1) $$x^2 = t_C(x)x - n_C(x)1_C,$$

(2) $$t_C(x) = n_C(1_C, x),$$

(3) $$n_C(1_C) = 1, \quad t_C(1_C) = 2,$$

(4) $$\bar{x} = t_C(x)1_C - x, \quad \bar{1}_C = 1_C, \quad \bar{\bar{x}} = x,$$

(5) $$x \circ y := xy + yx = t_C(x)y + t_C(y)x - n_C(x, y)1_C,$$

(6) $$x\bar{x} = n_C(x)1_C = \bar{x}x, \quad x + \bar{x} = t_C(x)1_C,$$

(7) $$n_C(\bar{x}) = n_C(x), \quad t_C(\bar{x}) = t_C(x),$$

(8) $$t_C(x^2) = t_C(x)^2 - 2n_C(x),$$

(9) $$t_C(x \circ y) = t_C(xy) + t_C(yx) = 2[t_C(x)t_C(y) - n_C(x, y)],$$

(10) $$n_C(x, \bar{y}) = t_C(x)t_C(y) - n_C(x, y),$$

(11) $$\overline{xy} - \bar{y}\bar{x} = \big(t_C(x, y) - n_C(x, \bar{y})\big)1_C.$$

2.3. **Norm-associative conic algebras.** Let C be a conic algebra over k as in 2.1. By [16, Prop. 18.10, (18.11.1)], the following (collections of) identities are all equivalent in C.

$$n_C(x, yx) = t_C(y)n_C(x), \tag{1}$$

$$n_C(x, xy) = t_C(y)n_C(x), \tag{2}$$

$$n_C(xy, z) = n_C(x, z\overline{y}), \tag{3}$$

$$n_C(xy, z) = n_C(y, \overline{x}z), \tag{4}$$

$$t_C(xy) = n_C(x, \overline{y}) = t_C(x)t_C(y) - n_C(x, y) = t_C(yx), \quad t_C\big((xy)z\big) = t_C\big(x(yz)\big). \tag{5}$$

If they are fulfilled, C is said to be *norm-associative*. By [16, Prop. 18.12], norm-associative conic algebras are flexible, and their conjugations are (algebra) involutions.

2.4. **Multiplicative conic algebras.** Following [16, 19.1], a conic algebra C over k is said to be *multiplicative* if its norm permits composition: $n_C(xy) = n_C(x)n_C(y)$ for all $x, y \in C$. By [16, 19.2 (a)], multiplicative conic algebras are norm-associative.

2.5. **Conic alternative algebras.** Let C be a conic alternative k-algebra, so C is a conic algebra satisfying the alternative laws. By [16, (19.3.2)], the U-operator of C satisfies the identity

$$U_x y = xyx = n_C(x, \bar{y})x - n_C(x)\bar{y}. \tag{1}$$

Note by [16, Exc. 68] that conic alternative algebras will in general not be multiplicative, though they are if the underlying module is projective [16, Prop. 19.5].

2.6. **The classical Cayley-Dickson construction.** Let B be any conic algebra over k and $\mu \in k$ an arbitrary scalar. We define a k-algebra C on the direct sum $B \oplus Bj$ of two copies of B as a k-module by the multiplication

$$(u_1 + v_1 j)(u_2 + v_2 j) := (u_1 u_2 + \mu \bar{v}_2 v_1) + (v_2 u_1 + v_1 \bar{u}_2)j,$$

for $u_i, v_i \in B$, $i = 1, 2$, and a quadratic form $n_C\colon C \to k$ by

$$n_C(u + vj) := n_B(u) - \mu n_B(v) \qquad (u, v \in B).$$

C together with n_C is a conic k-algebra and is said to arise from B, μ by means of the *Cayley-Dickson construction*, written as $\mathrm{Cay}(B, \mu)$ in order to indicate dependence on the parameters involved. Note that $1_C = 1_B + 0 \cdot j$ is an identity element for C and that the assignment $u \mapsto u + 0 \cdot j$ gives an *embedding*, i.e., an injective homomorphism, $B \hookrightarrow C$ of conic k-algebras, allowing us to identify $B \subseteq C$ as a conic subalgebra.

3. Elementary properties of the non-orthogonal Cayley-Dickson construction

To the best of our knowledge, the first examples of a non-orthogonal Cayley-Dickson construction are due to Pumplün [17] and work over an integral domain whose quotient field has characteristic not 2 by using bases and structure constants closely modeled after the Coxeter octonions [2]. Roughly speaking, the non-orthogonal Cayley-Dickson construction proposed in the present paper formalizes the most general way in which a given conic algebra sits in a multiplicative alternative one as a unital subalgebra. The details of this formalization may be read off from the following observation.

3.1. **Proposition.** (The internal construction) *Let C be a multiplicative conic alternative k-algebra, B a unital subalgebra of C and $l \in C$. Then the subalgebra C' of C generated by B and l agrees with $B + Bl$ as a k-submodule. More precisely, writing $s\colon B \to k$ for the linear form defined by*

$$s(u) := n_C(u, l) \qquad (u \in B) \tag{1}$$

and setting

$$\lambda := s(1_C) = t_C(l) \in k, \quad \mu := -n_C(l) \in k, \tag{2}$$

the relations

$$(vl)u = -s(u)v + \lambda vu + (v\bar{u})l, \tag{3}$$

$$u(vl) = -s(\bar{v}u)1_B + s(u)v + s(\bar{v})u - \lambda vu + (vu)l, \tag{4}$$

$$u \circ (vl) = -s(\bar{v}u)1_B + s(\bar{v})u + t_B(u)vl, \tag{5}$$

$$\begin{aligned}(v_1l)(v_2l) = &\big(-\lambda s(\bar{v}_2v_1)1_B + \lambda s(v_1)v_2 + \lambda s(\bar{v}_2)v_1 - \lambda^2 v_2v_1 + \mu\bar{v}_2v_1\big) \\ &+ \big(s(\bar{v}_2v_1)1_B - s(v_1)v_2 + \lambda v_2v_1\big)l,\end{aligned} \tag{6}$$

$$n_C(u + vl) = n_B(u) + s(\bar{v}u) - \mu n_B(v) \tag{7}$$

hold for all $u, u_1, u_2, v, v_1, v_2 \in B$.

Proof. Firstly, we simplify notation by writing $n := n_C$ (resp. $t := t_C$) for the norm (resp. trace) of C. Secondly, we note that the first assertion will follow once we have established the identities (3)−(7). In order to do so, we begin with (5) and, applying (2.2.5), (2.2.7), (2.3.4), (2.3.5), (1), obtain $u \circ (vl) = t(u)(vl) + t(vl)u - n(u, vl)1_B = t(u)(vl) + n(\bar{v}, l)u - n(\bar{v}u, l)1_B = t(u)(vl) + s(\bar{v})u - s(\bar{v}u)1_B$, giving (5). Next we use right alternativity linearized to compute $(vl)u+(vu)l = v(l\circ u) = t(l)vu+t(u)vl-n(u, l)v$, which implies

$$(vl)u = t(l)vu + \big(v\big[t(u)1_B - u\big]\big)l - n(u, l)v = -s(u)v + \lambda vu + (v\bar{u})l,$$

hence (3). Subtracting (3) from (5) gives (4). Now we proceed to derive (6). Combining (5) with the middle Moufang identity and (2.5.1), we obtain

$$\begin{aligned}(v_1l)(v_2l) &= (v_1 \circ l)(v_2l) - (lv_1)(v_2l) = \big(-s(v_1)1_B + \lambda v_1 + t(v_1)l\big)(v_2l) - l(v_1v_2)l \\ &= t(v_1)lv_2l + \lambda v_1(v_2l) - s(v_1)(v_2l) - l(v_1v_2)l \\ &= l(\bar{v}_1v_2)l + \lambda v_1(v_2l) - s(v_1)(v_2l) \\ &= n(l, \bar{v}_2v_1)l - n(l)\bar{v}_2v_1 \\ &\quad + \lambda\big(-s(\bar{v}_2v_1)1_B + s(v_1)v_2 + s(\bar{v}_2)v_1 - \lambda v_2v_1 + (v_2v_1)l\big) - s(v_1)(v_2l) \\ &= s(\bar{v}_2v_1)l - s(v_1)(v_2l) + \mu\bar{v}_2v_1 \\ &\quad - \lambda s(\bar{v}_2v_1)1_B + \lambda s(v_1)v_2 + \lambda s(\bar{v}_2)v_1 - \lambda^2 v_2v_1 + \lambda(v_2v_1)l,\end{aligned}$$

and this is (6). It remains to establish (7). This follows immediately from multiplicativity, (2.3.4), (1), (2) and the expansion $n(u+vl) = n(u)+n(u, vl)+n(v)n(l) = n(u)+s(\bar{v}u)-\mu n(v)$. □

3.2. **The external construction.** Let B be a conic k-algebra, $s\colon B \to k$ an arbitrary linear form and $\mu \in k$ an arbitrary scalar. Motivated by the formulas derived in Prop. 3.1, we put

$$\lambda := s(1_B) \tag{1}$$

and write

$$C := \mathrm{Cay}(B; \mu, s) := B \oplus Bj \tag{2}$$

for the non-associative k-algebra living on the direct sum of two copies of B as a k-module under a bilinear multiplication uniquely determined by the condition that the k-module B, identified in C through the initial summand, is a subalgebra and the equations

$$u(vj) = \big(-s(\bar{v}u)1_B + s(u)v + s(\bar{v})u - \lambda vu\big) + (vu)j, \tag{3}$$

$$(vj)u = \big(-s(u)v + \lambda vu\big) + (v\bar{u})j, \tag{4}$$

$$\begin{aligned}(v_1j)(v_2j) = &\Big(-\lambda s(\bar{v}_2v_1)1_B + \lambda s(v_1)v_2 + \lambda s(\bar{v}_2)v_1 - \lambda^2 v_2v_1 + \mu\bar{v}_2v_1\Big) \\ &+ \Big(s(\bar{v}_2v_1)1_B - s(v_1)v_2 + \lambda v_2v_1\Big)j\end{aligned} \tag{5}$$

hold for all $u, v, v_1, v_2 \in B$. One checks immediately that the k-algebra C is unital, with unit element $1_C := 1_B$, so the subalgebra $B \subseteq C$ is, in fact, unital.

3.3. **Proposition.** *Under the assumptions and notation of* 3.2, $C = \mathrm{Cay}(B; \mu, s)$ *is a conic k-algebra with unit element, norm, bilinearized norm, trace, conjugation respectively given by*

$$1_C = 1_B, \tag{1}$$

$$n_C(u + vj) = n_B(u) + s(\bar{v}u) - \mu n_B(v), \tag{2}$$

$$n_C(u_1 + v_1 j, u_2 + v_2 j) = n_B(u_1, u_2) + s(\bar{v}_2 u_1) + s(\bar{v}_1 u_2) - \mu n_B(v_1, v_2), \tag{3}$$

$$t_C(u + vj) = t_B(u) + s(\bar{v}), \tag{4}$$

$$\overline{u + vj} = \bar{u} + s(\bar{v})1_B - vj \tag{5}$$

for all $u, u_1, u_2, v, v_1, v_2 \in B$. We also have

$$u \circ (vj) = \big(- s(\bar{v}u)1_B + s(\bar{v})u\big) + \big(t_B(u)v\big)j \tag{6}$$

for all $u, v \in B$.

Proof. We have seen already in 3.2 that C is unital and (1) holds. Moreover, (2) defines a quadratic form n_C on C whose bilinearization is given by (3), and $t_C := n_C(1_C, -)$ satisfies (4). Adding (3.2.3) to (3.2.4), we arrive at (6). Hence, using (3.2.5), we may compute

$$\begin{aligned}
(u + vj)^2 &= u^2 + u \circ (vj) + (vj)^2 \\
&= t_B(u)u - n_B(u)1_B - s(\bar{v}u)1_B + s(\bar{v})u + \big(t_B(u)v\big)j - \lambda^2 n_B(v)1_B \\
&\quad + \lambda^2 t_B(v)v - \lambda^2 v^2 + \mu n_B(v)1_B + \big(\lambda n_B(v)1_B - s(v)v + \lambda v^2\big)j \\
&= t_B(u)(u + vj) + s(\bar{v})u + \big(\lambda t_B(v) - s(v)\big)(vj) \\
&\quad - \big(n_B(u) + s(\bar{v}u) - \mu n_B(v)\big)1_B \\
&= \big(t_B(u) + s(\bar{v})\big)(u + vj) - n_C(u + vj)1_C \\
&= t_C(u + vj)(u + vj) - n_C(u + vj)1_C.
\end{aligned}$$

Thus C is a conic k-algebra with norm, bilinearized norm and trace as indicated. Finally, the formula for the conjugation of C is now obvious. $\square$

3.4. **Examples.** In the situation of 3.2, 3.3, the conic algebra $\mathrm{Cay}(B; \mu, s)$ over k is said to arise from B, μ, s by the *non-orthogonal Cayley-Dickson construction*. If $s = 0$ is the zero linear form, formulas (3.2.3)−(3.2.5) combined with (3.3.2) show that the general non-orthogonal Cayley-Dickson construction collapses to the ordinary one: $\mathrm{Cay}(B; \mu, 0) = \mathrm{Cay}(B, \mu)$.

On the other hand, suppose K/k is a purely inseparable field extension of characteristic 2 and exponent at most 1, and suppose further $s\colon K \to k$ is a linear form which is unital in the sense that $s(1_K) = 1$. Since K as a conic k-algebra has trivial conjugation, a comparison of (3.2.3)−(3.2.5) and (3.3.2) with [5, (4.3.1)−(4.3.3), (4.4.1)] shows that our general non-orthogonal Cayley-Dickson construction collapses to the one investigated in [5, §§4,5].

3.5. **Proposition.** *Let C be a multiplicative alternative conic k-algebra, B a conic k-algebra and $\varphi\colon B \to C$ a homomorphism of conic algebras. Given an element $l \in C$, put $\mu := -n_C(l) \in k$ and define a linear form $s\colon B \to k$ by*

$$s(u) := n_C\big(\varphi(u), l\big) \qquad (u \in B). \tag{1}$$

Then there is a unique extension of φ to a homomorphism $\varphi'\colon \mathrm{Cay}(B; \mu, s) = B \oplus Bj \to C$ of conic k-algebras such that $\varphi'(j) = l$.

Proof. Uniqueness follows from the obvious fact that the conic k-algebra $C' := \mathrm{Cay}(B; \mu, s)$ is generated by B and j. To prove existence, we define $\varphi'\colon C' \to C$ by

$\varphi'(u+vj) = \varphi(u) + \varphi(v)l$ for all $u, v \in B$. Then φ' is a k-linear map extending φ, and we must show

$$\varphi'(u(vj)) = \varphi(u)(\varphi(v)l),$$
$$\varphi'((vj)u) = (\varphi(v)l)\varphi(u),$$
$$\varphi'((v_1j)(v_2j)) = (\varphi(v_1)l)(\varphi(v_2)l),$$

and, finally,

$$n_C(\varphi(u) + \varphi(v)l) = n_{C'}(u+vj)$$

for all $u, v, v_1, v_2 \in B$. But setting $\lambda := s(1_B) = t_C(l)$, these formulas follow immediately by comparing (3.2.3)−(3.2.5) and (3.3.2) with the relations (3.1.3)−(3.1.7). □

3.6. **Corollary.** *Let B be a conic k-algebra, $\mu \in k$ an arbitrary scalar and $s\colon B \to k$ an arbitrary linear form. Put $\lambda := s(1_B)$, $C := \mathrm{Cay}(B; \mu, s) = B \oplus Bj$ as in* 3.2 *and, given elements $a, b \in B$, define a scalar*

$$\mu' := -n_B(a) - s(\bar{b}a) + \mu n_B(b) \in k \tag{1}$$

as well as a linear form $s'\colon B \to k$ by

$$s'(u) := n_B(u, a) + s(\bar{b}u) \qquad (u \in B). \tag{2}$$

If C is multiplicative alternative, then the linear map $\varphi\colon C' := \mathrm{Cay}(B; \mu', s') = B \oplus Bj' \to C$ defined by

$$\varphi(u + vj') := \big(u + va - s(\bar{b}v)1_B + s(v)b + s(\bar{b})v - \lambda bv\big) + (bv)j \tag{3}$$

is a homomorphism of conic algebras; moreover, for $b \in B^\times$, φ is an isomorphism of conic algebras.

Proof. φ extends the identity of B and satisfies the relation $\varphi(j') = l := a + bj$. By (3.3.2), (1), (3.3.3), (2) we have $-n_C(l) = -n_C(a+bj) = -n_B(a) - s(\bar{b}a) + \mu n_B(b) = \mu'$, $n_C(u, l) = n_C(u, a+bj) = n_B(u, a) + s(\bar{b}u) = s'(u)$ for all $u \in B$. Hence Prop. 3.5 yields a unique homomorphism $\psi\colon C' \to C$ extending the identity of B and sending j' to l. For all $u, v \in B$ we may apply (3.2.3) and (3) to obtain

$$\begin{aligned}\psi(u + vj') &= u + vl = u + va + v(bj)\\ &= \big(u + va - s(\bar{b}v)1_B + s(v)b + s(\bar{b})v - \lambda bv\big) + (bv)j\\ &= \varphi(u + vj').\end{aligned}$$

Hence $\varphi = \psi$ is a homomorphism of conic algebras. Now assume $b \in B^\times$. Then (3) shows that φ is injective. Its image contains B and l, hence also $j = b^{-1}(-a + l)$. But the algebra C is generated by B and j, forcing φ to be surjective as well. □

4. The conjugation, norm-associativity and flexibility

In the first part of this section we generalize [16, Prop. 20.10] by showing that the property of the conjugation of a conic algebra to be an involution is preserved by the non-orthogonal Cayley-Dickson construction. Our approach is based on the following concept.

4.1. **A peculiar linear form.** Let C be a conic algebra over k. We define a linear form $m_C\colon C \otimes C \to k$ by

$$m_C(x \otimes y) := t_C(xy) - n_C(x, \bar{y}) \qquad (x, y \in C). \tag{1}$$

By [16, Prop. 18.8], the conjugation of C is an involution if and only if $\mathrm{Im}(m_C) \subseteq \mathrm{Ann}(C)$.

4.2. **Linear forms on conic algebras.** Let B be a conic algebra over k and $s\colon B \to k$ any linear form. If we put $\lambda := s(1_B)$, then (2.2.1), (2.2.5), (2.2.4) imply

$$s(u^2) = t_B(u)s(u) - \lambda n_B(u), \tag{1}$$

$$s(uv) + s(vu) = t_B(u)s(v) + t_B(v)s(u) - \lambda n_B(u, v), \tag{2}$$

$$s(\bar{u}) = \lambda t_B(u) - s(u) \tag{3}$$

for all $u, v \in B$.

4.3. **Proposition.** *Let B be a conic k-algebra, $\mu \in k$ and $s\colon B \to k$ a linear form. Setting $C := \mathrm{Cay}(B; \mu, s) = B \oplus Bj$ as in 3.2, we have*

$$t_C\big((vj)u\big) = n_C(vj, \bar{u}), \tag{1}$$

$$t_C\big(u(vj)\big) = n_C(u, \overline{vj}), \tag{2}$$

$$t_C\big((v_1 j)(v_2 j)\big) = n_C(v_1 j, \overline{v_2 j}) + \mu\big(t_B(\bar{v}_2 v_1) - n_B(\bar{v}_2, \bar{v}_1)\big) \tag{3}$$

for all $u, v, v_1, v_2 \in B$.

Proof. Beginning with (1), we apply (3.2.4), (3.3.4), (4.2.3), (2.2.6), (3.3.3) to obtain

$$\begin{aligned} t_C\big((vj)u\big) &= t_C\Big(-\big(s(u)v + \lambda vu\big) + (v\bar{u})j\Big) \\ &= -s(u)t_B(v) + \lambda t_B(vu) + s(\overline{v\bar{u}}) \\ &= -s(u)t_B(v) + \lambda t_B(vu) + \lambda t_B(v\bar{u}) - s(v\bar{u}) \\ &= -s(u)t_B(v) + \lambda t_B(u)t_B(v) - t_B(u)s(v) + s(vu) \\ &= s\Big(\big(t_B(v)1_B - v\big)\big(t_B(u)1_B - u\big)\Big) \\ &= s(\bar{v}\,\bar{u}) = n_C(vj, \bar{u}), \end{aligned}$$

as desired. Next we reduce (2) to (1) by setting $x = u$, $y = vj$ and applying (2.2.9), (1), (2.2.10), (2.2.7). Then

$$\begin{aligned} t_C(xy) &= t_C(x \circ y) - t_C(yx) = 2\big(t_C(x)t_C(y) - n_C(x, y)\big) - n_C(y, \bar{x}) \\ &= 2n_C(x, \bar{y}) - n_C(x, \bar{y}) = n_C(x, \bar{y}), \end{aligned}$$

and we have (2). Finally, turning to (3) and applying (3.2.5), (2.2.3), (3.3.4), (4.2.3), (2.2.10), we obtain

$$\begin{aligned} t_C\big((v_1 j)(v_2 j)\big) &= t_C\Big(\big(-\lambda s(\bar{v}_2 v_1)1_B + \lambda s(v_1)v_2 + \lambda s(\bar{v}_2)v_1 - \lambda^2 v_2 v_1 + \mu \bar{v}_2 v_1\big) \\ &\quad + \big(s(\bar{v}_2 v_1)1_B - s(v_1)v_2 + \lambda v_2 v_1\big)j\Big) \\ &= -2\lambda s(\bar{v}_2 v_1) + \lambda s(v_1)t_B(v_2) + \lambda s(\bar{v}_2)t_B(v_1) - \lambda^2 t_B(v_2 v_1) \\ &\quad + \mu t_B(\bar{v}_2 v_1) + \lambda s(\bar{v}_2 v_1) - s(v_1)s(\bar{v}_2) + \lambda s(\overline{\bar{v}_2 v_1}) \\ &= -\lambda t_B(v_2)s(v_1) + \lambda s(v_2 v_1) + \lambda t_B(v_2)s(v_1) + \lambda^2 t_B(v_1)t_B(v_2) \\ &\quad - \lambda t_B(v_1)s(v_2) - \lambda^2 t_B(v_2 v_1) + \mu t_B(\bar{v}_2 v_1) - \lambda t_B(v_2)s(v_1) \\ &\quad + s(v_1)s(v_2) + \lambda^2 t_B(v_2 v_1) - \lambda s(v_2 v_1) \\ &= -\lambda t_B(v_2)s(v_1) + \lambda^2 t_B(v_1)t_B(v_2) - \lambda t_B(v_1)s(v_2) + \mu t_B(\bar{v}_2 v_1) \\ &\quad + s(v_1)s(v_2). \end{aligned}$$

On the other hand, by (3.3.5), (3.3.3), (4.2.3),

$$\begin{aligned} n_C(v_1 j, \overline{v_2 j}) &= n_C\big(v_1 j, s(\bar{v}_2)1_B - v_2 j\big) = s\big(\bar{v}_1 s(\bar{v}_2)1_B\big) + \mu n_B(v_1, v_2) \\ &= s(\bar{v}_1)s(\bar{v}_2) + \mu n_B(v_1, v_2) \\ &= \big(\lambda t_B(v_1) - s(v_1)\big)\big(\lambda t_B(v_2) - s(v_2)\big) + \mu n_B(v_1, v_2) \\ &= \lambda^2 t_B(v_1)t_B(v_2) - \lambda t_B(v_1)s(v_2) - \lambda t_B(v_2)s(v_1) + s(v_1)s(v_2) \\ &\quad + \mu n_B(v_1, v_2), \end{aligned}$$

and a comparison with the preceding equation yields (3). ☐

4.4. **Corollary.** $\mathrm{Im}(m_B) = \mathrm{Im}(m_C)$.

Proof. Since B is a direct summand of C as a k-module, $B \otimes B$ may be viewed canonically as a submodule of $C \otimes C$, and m_B is the restriction of m_C to $B \otimes B$. Now, by 4.1 and Prop. 4.3, the ideal $\mathrm{Im}(m_C) \subseteq k$ is generated by the expressions

$$m_B(u \otimes v), \quad m_C\big((vj) \otimes u\big) = 0, \quad m_C\big(u \otimes (vj)\big) = 0,$$
$$m_C\big((v_1 j) \otimes (v_2 j)\big) = \mu m_B\big(\bar{v}_2 \otimes v_1\big)$$

for all $u, v, v_1, v_2 \in B$. The assertion follows. ☐

4.5. **Corollary.** *If the conjugation of B is an involution, then so is the conjugation of C.*

Proof. Since B and C have the same annihilator, this follows immediately from 4.1 and Cor. 4.4. ☐

Next we wish to describe conditions under which the property of a conic algebra to be norm-associative is preserved by the non-orthogonal Cayley-Dickson construction. To this end, we need a conceptual preparation.

4.6. **Reminder: associative linear forms.** Let A be a non-associative k-algebra. A linear form $t\colon A \to k$ is said to be *associative* if $t((xy)z) = t(x(yz))$ for all $x, y, z \in A$, equivalently, if t vanishes on all associators of A: $t([A, A, A]) = \{0\}$. Clearly, a linear form on an associative algebra is automatically associative. On the other hand, if A is arbitrary but k is a field, non-zero associative linear forms on A exist if and only if $[A, A, A] \subset A$ is a proper subspace.

4.7. **Alternative linear forms.** We continue to consider an arbitrary non-associative k-algebra A. A linear form $s\colon A \to k$ is called *alternative* if the trilinear map

$$A \times A \times A \longrightarrow k, \quad (x, y, z) \longmapsto s([x, y, z])$$

is alternating, equivalently, if two of the following relations

$$(1) \qquad s\big(x(xy)\big) = s(x^2 y), \quad s\big((yx)x\big) = s(yx^2), \quad s\big((xy)x\big) = s\big(x(yx)\big)$$

hold for all $x, y \in A$, in which case the third one follows automatically. Clearly, a linear form on an alternative algebra is always alternative. On the other hand, if A is arbitrary but k is a field, non-zero alternative linear forms on A exist if and only if the expressions $[x, x, y]$, $[y, x, x]$ for $x, y \in A$ span a proper subspace of A.

4.8. **Proposition.** *For a norm-associative conic algebra B over k and a linear form $s\colon B \to k$, the following conditions are equivalent.*

(i) *s is alternative.*
(ii) *$s(u(uv)) = t_B(u)s(uv) - n_B(u)s(v)$ for all $u, v \in B$.*
(iii) *$s((vu)u) = t_B(u)s(vu) - n_B(u)s(v)$ for all $u, v \in B$.*
(iv) *$s(uvu) = n_B(u, \bar{v})s(u) - n_B(u)s(\bar{v})$ for all $u, v \in B$.*

Proof. Since B is flexible by [16, Prop. 18.12], the third equation of (4.7.1) holds. Hence either one of the first two is equivalent to s being alternative. Combined with (2.2.1), this shows that conditions (i), (ii), (iii) are equivalent. It remains to establish the implications (ii)⇒(iv)⇒(iii).

(ii) ⇒ (iv). Setting $\lambda := s(1_B)$ and combining (ii) with (4.2.2), (2.3.5), (2.3.2), we obtain

$$\begin{aligned} s(uvu) = s\big((uv)u\big) &= t_B(uv)s(u) + t_B(u)s(uv) - \lambda n_B(uv, u) - s\big(u(uv)\big) \\ &= t_B(uv)s(u) + n_B(u)s(v) - \lambda t_B(v)n_B(u) \\ &= n_B(u, \bar{v})s(u) + n_B(u)\big(s(v) - \lambda t_B(v)\big), \end{aligned}$$

and (4.2.3) yields (iv).

(iv) $\Rightarrow$ (iii). Using (iv) and (2.2.5), (4.2.1), (4.2.3), we compute

$$\begin{aligned}
s\big((vu)u\big) &= s\big((u\circ v)u\big) - s(uvu)\\
&= t_B(u)s(vu) + t_B(v)s(u^2) - n_B(u,v)s(u) - n_B(u,\bar v)s(u) + n_B(u)s(\bar v)\\
&= t_B(u)s(vu) + t_B(u)t_B(v)s(u) - \lambda n_B(u)t_B(v) - t_B(u)t_B(v)s(u) + n_B(u)s(\bar v)\\
&= t_B(u)s(vu) - n_B(u)s(v),
\end{aligned}$$

and this is (iii). ☐

4.9. **Lemma.** *Let B be a norm-associative conic k-algebra, $s\colon B \to k$ an alternative linear form and $\lambda := s(1_B)$. Then*

$$s\big(u(vw)\big) + s\big((wu)v\big) = s(w)t_B(uv) + t_B(w)s(uv) - \lambda n_B(uv,w), \tag{1}$$

$$\begin{aligned}
s\big((uv)w\big) + s\big(u(vw)\big) = {}& t_B(u)s(vw) - t_B(v)s(wu) + t_B(w)s(uv)\\
&+ t_B(vw)s(u) - t_B(wu)s(v) + t_B(uv)s(w)\\
&+ \lambda\big(n_B(uw,v) - t_B(v)n_B(u,w)\big)
\end{aligned}\tag{2}$$

for all $u,v,w\in B$.

Proof. Applying (4.2.2), (2.3.5) and Prop. 4.8, we obtain

$$\begin{aligned}
s\big(u(vw)\big) + s\big((wu)v\big) &= s\big(u(vw)\big) + s\big((wu)v\big) + s\big(v(wu)\big) - s\big(v(wu)\big)\\
&= s\big(u(vw)\big) + t_B(wu)s(v) + t_B(v)s(wu) - \lambda n_B(wu,v)\\
&\quad - s\big(v(w\circ u)\big) + s\big(v(uw)\big)\\
&= t_B(wu)s(v) + t_B(v)s(wu) - \lambda n_B(wu,v) - t_B(w)s(vu)\\
&\quad - t_B(u)s(vw) + n_B(w,u)s(v) + s\big(u(vw) + v(uw)\big)\\
&= \big(t_B(wu) + n_B(w,u)\big)s(v) + t_B(v)s(wu) - \lambda n_B(wu,v)\\
&\quad - t_B(w)s(vu) - t_B(u)s(vw) + t_B(u)s(vw)\\
&\quad + t_B(v)s(uw) - n_B(u,v)s(w)\\
&= t_B(w)t_B(u)s(v) + t_B(v)s(uw+wu) - \lambda n_B(wu,v)\\
&\quad - t_B(w)s(uv+vu) + t_B(w)s(uv) - n_B(u,v)s(w)\\
&= t_B(w)t_B(u)s(v) + t_B(v)t_B(u)s(w) + t_B(v)t_B(w)s(u)\\
&\quad - \lambda t_B(v)n_B(u,w) - \lambda n_B(wu,v) - t_B(w)t_B(u)s(v)\\
&\quad - t_B(w)t_B(v)s(u) + \lambda t_B(w)n_B(u,v) + t_B(w)s(uv)\\
&\quad - n_B(u,v)s(w)\\
&= t_B(uv)s(w) + t_B(w)s(uv)\\
&\quad - \lambda\big(n_B(wu,v) + t_B(v)n_B(u,w) - t_B(w)n_B(u,v)\big).
\end{aligned}$$

Using (2.3.3), (2.3.4), (2.2.4), we can now compute

$$\begin{aligned}
n_B(wu,v) = n_B(u,\bar w v) = n_B(u\bar v,\bar w) &= n_B\Big(u\big(t_B(v)1_B - v\big), t_B(w)1_B - w\Big)\\
&= t_B(u)t_B(v)t_B(w) - t_B(w)t_B(uv) - t_B(v)n_B(u,w) + n_B(uv,w)\\
&= t_B(w)n_B(u,v) - t_B(v)n_B(u,w) + n_B(uv,w).
\end{aligned}$$

Inserting this into the factor of λ in the final expression of the preceding equation, we end up with (1). Turning to (2) and combining Prop. 4.8 with (4.2.2), we obtain

$$\begin{aligned} s\big((uv)w\big) &= s\big((uv)w + (uw)v\big) - s\big((uw)v\big) \\ &= t_B(v)s(uw) + t_B(w)s(uv) - n_B(v,w)s(u) - s\big((uw)\circ v\big) + s\big(v(uw)\big) \\ &= t_B(v)s(uw) + t_B(w)s(uv) - n_B(v,w)s(u) - t_B(uw)s(v) - t_B(v)s(uw) \\ &\quad + \lambda n_B(uw,v) + s\big(v(uw) + u(vw)\big) - s\big(u(vw)\big) \\ &= t_B(w)s(uv) - n_B(v,w)s(u) - t_B(uw)s(v) + \lambda n_B(uw,v) + t_B(u)s(vw) \\ &\quad + t_B(v)s(uw) - n_B(u,v)s(w) - s\big(u(vw)\big). \end{aligned}$$

By (4.2.2) again, the sixth summand on the right agrees with

$$\begin{aligned} t_B(v)s(uw) &= t_B(v)s(uw + wu) - t_B(v)s(wu) \\ &= t_B(u)t_B(v)s(w) + t_B(v)t_B(w)s(u) - \lambda t_B(v)n_B(w,u) - t_B(v)s(wu). \end{aligned}$$

Returning with this to the preceding equation, we conclude

$$\begin{aligned} s\big((uv)w\big) &= t_B(w)s(uv) - n_B(v,w)s(u) - t_B(uw)s(v) + \lambda n_B(uw,v) + t_B(u)s(vw) \\ &\quad + t_B(u)t_B(v)s(w) + t_B(v)t_B(w)s(u) - \lambda t_B(v)n_B(w,u) \\ &\quad - t_B(v)s(wu) - n_B(u,v)s(w) - s\big(u(vw)\big) \\ &= t_B(u)s(vw) - t_B(v)s(wu) + t_B(w)s(uv) \\ &\quad + t_B(vw)s(u) - t_B(wu)s(v) + t_B(uv)s(w) \\ &\quad + \lambda\big(n_B(uw,v) - t_B(v)n_B(u,w)\big) - s\big(u(vw)\big), \end{aligned}$$

and (2) follows. ☐

4.10. **Theorem.** *Let B be a conic k-algebra, $\mu \in k$ a scalar and $s\colon B \to k$ a linear form. Then the conic algebra $\mathrm{Cay}(B;\mu,s) = B \oplus Bj$ is norm-associative if and only if B is norm-associative and s is alternative.*

Proof. If $C := \mathrm{Cay}(B;\mu,s)$ is norm-associative, so is $B \subseteq C$ (as a unital subalgebra). Moreover, given $u, v \in B$, we apply (3.3.3), (2.3.4), (3.2.3), (2.2.1) and compute

$$\begin{aligned} s\big(u(uv)\big) &= n_C\big(u(uv), j\big) = n_C\big(v, \bar u(\bar u j)\big) \\ &= n_C\big(v, -s(u\bar u)1_B + s(\bar u)\bar u + s(u)\bar u - \lambda \bar u^2 + \bar u^2 j\big) \\ &= n_C\big(v, -\lambda n_B(\bar u)1_B + \lambda t_B(\bar u)\bar u - \lambda \bar u^2 + \bar u^2 j\big) \\ &= n_C(v, \bar u^2 j) = n_C(u^2 v, j) = s(u^2 v) = t_B(u)s(uv) - n_B(u)s(v). \end{aligned}$$

Thus condition (ii) of Prop. 4.8 holds, forcing s to be alternative.

Conversely, suppose B is norm-associative and s is alternative. Then B is flexible, the conjugation of B is an involution [16, Prop. 18.12], equations (2.3.1)−(2.3.5) hold in B, and by 2.3 it will be enough to show that (2.3.2) holds in C. Actually, by linearity, it suffices to establish the relations

$$n_C\big((u+vj)w, u+vj\big) = n_C(u+vj)t_B(w), \tag{1}$$

$$n_C\big((u+vj)(wj), u+vj\big) = n_C(u+vj)t_C(wj) \tag{2}$$

for all $u, v, w \in B$. Linearizing we see that (2) is equivalent to the following three identities.

$$n_C\big(u(wj), u\big) = n_B(u)t_C(wj), \tag{3}$$

$$n_C\big(u(wj), vj\big) + n_C\big((vj)(wj), u\big) = n_C(u, vj)t_C(wj), \tag{4}$$

$$n_C\big((vj)(wj), vj\big) = n_C(vj)t_C(wj), \tag{5}$$

again for all $u, v, w \in B$.

We begin with the verification of (1). Applying (3.2.4), (3.3.2), (3.3.3), we obtain

$$\begin{aligned}
&n_C\big((u{+}vj)w, u+vj\big) - n_C(u+vj)t_B(w) \\
&\quad = n_C\big([uw - s(w)v + \lambda vw] + (v\bar{w})j, u+vj\big) - \big[n_B(u) + s(\bar{v}u) - \mu n_B(v)\big]t_B(w) \\
&\quad = n_B(uw,u) - s(w)n_B(v,u) + \lambda n_B(vw,u) + s\big(\bar{v}(uw)\big) - s(w)s(\bar{v}v) \\
&\qquad + \lambda s\big(\bar{v}(vw)\big) + s\big((w\bar{v})u\big) - \mu n_B(v\bar{w},v) - n_B(u)t_B(w) \\
&\qquad - s(\bar{v}u)t_B(w) + \mu n_B(v)t_B(w)
\end{aligned}$$

Here norm-associativity of B by (2.3.2) implies

$$n_B(uw,u) = n_B(u)t_B(w), \quad n_B(v\bar{w},v) = n_B(v)t_B(\bar{w}) = n_B(v)t_B(w).$$

On the other hand, we trivially have $s(w)s(\bar{v}v) = \lambda n_B(v)s(w)$, while Prop. 4.8 yields $s(\bar{v}(vw)) = t_B(v)s(vw) - s(v(vw)) = n_B(v)s(w)$ since s is alternative. Finally, we deduce from (4.9.1), (2.3.4)

$$\begin{aligned}
s\big(\bar{v}(uw)\big) + s\big((w\bar{v})u\big) &= s(w)t_B(\bar{v}u) + t_B(w)s(\bar{v}u) - \lambda n_B(\bar{v}u,w) \\
&= s(w)n_B(v,u) + t_B(w)s(\bar{v}u) - \lambda n_B(u,vw).
\end{aligned}$$

Inserting all this into the final expression of the displayed chain of equations above, it follows that this expression is zero, hence that (1) holds.

We now proceed to deduce (3) by using (3.2.3), (3.3.4), (2.2.4) and Prop. 4.8 (iv) combined with (2.3.2), which imply

$$\begin{aligned}
&n_C\big(u(wj),u\big) - n_B(u)t_C(wj) \\
&\quad = n_C\Big(\big[-s(\bar{w}u)1_B + s(u)w + s(\bar{w})u - \lambda wu\big] + (wu)j, u\Big) - n_B(u)s(\bar{w}) \\
&\quad = -s(\bar{w}u)t_B(u) + s(u)n_B(w,u) + 2s(\bar{w})n_B(u) - \lambda n_B(wu,u) \\
&\qquad + s\big((\bar{u}\bar{w})u\big) - n_B(u)s(\bar{w}) \\
&\quad = -s(\bar{w}u)t_B(u) + s(u)n_B(w,u) + s(\bar{w})n_B(u) - \lambda n_B(wu,u) \\
&\qquad + t_B(u)s(\bar{w}u) - n_B(u,w)s(u) + n_B(u)s(w) \\
&\quad = \lambda t_B(w)n_B(u) - \lambda n_B(wu,u) = 0,
\end{aligned}$$

and the proof of (3) is complete.

We now come to the verification of (4), which is the most involved one of all. We begin by using (3.2.3), (3.2.5), (3.3.3) to manipulate the left-hand side as follows:

$$\begin{aligned}
&n_C\big(u(wj),vj\big) + n_C\big((vj)(wj),u\big) \\
&\quad = n_C\Big(\big[-s(\bar{w}u)1_B + s(u)w + s(\bar{w})u - \lambda wu\big] + (wu)j, vj\Big) \\
&\qquad + n_C\Big(\big[-\lambda s(\bar{w}v)1_B + \lambda s(v)w + \lambda s(\bar{w})v - \lambda^2 wv + \mu\bar{w}v\big] \\
&\qquad + \big[s(\bar{w}v)1_B - s(v)w + \lambda wv\big]j, u\Big) \\
&\quad = -s(\bar{w}u)s(\bar{v}) + s(u)s(\bar{v}w) + s(\bar{w})s(\bar{v}u) - \lambda s\big(\bar{v}(wu)\big) - \mu n_B(wu,v) \\
&\qquad - \lambda s(\bar{w}v)t_B(u) + \lambda s(v)n_B(w,u) + \lambda s(\bar{w})n_B(v,u) - \lambda^2 n_B(wv,u) + \mu n_B(\bar{w}v,u) \\
&\qquad + s(\bar{w}v)s(u) - s(v)s(\bar{w}u) + \lambda s\big((\bar{v}\bar{w})u\big).
\end{aligned}$$

Here $\mu n_B(\bar{w}v,u) = \mu n_B(v,wu)$ cancels against the fifth summand on the very right of the preceding expression. Moreover, $-s(\bar{w}u)s(\bar{v}) - s(v)s(\bar{w}u) = -\lambda s(\bar{w}u)t_B(v)$, $s(u)s(\bar{v}w) + s(\bar{w}v)s(u) = \lambda s(u)t_B(\bar{v}w) = \lambda s(u)n_B(v,w)$, while $s(\bar{w})s(\bar{v}u) = n_C(u,vj)t_C(wj)$. Hence

$$n_C\big(u(wj),vj\big) + n_C\big((vj)(wj),u\big) = n_C(u,vj)t_C(wj) + \lambda\alpha,$$

where

$$\begin{aligned}\alpha := &-s(\bar{w}u)t_B(v)+s(u)n_B(v,w)-s(\bar{w}v)t_B(u)+s(v)n_B(w,u)+s(\bar{w})n_B(v,u)\\ &-\lambda n_B(wv,u)+s(\bar{v}u)t_B(w)-\Big(s\big((\bar{v}w)u\big)+s\big(\bar{v}(wu)\big)\Big)\end{aligned}$$

It suffices to show $\alpha=0$. To this end, we apply (4.9.2), (2.2.4), (2.3.2), (2.3.4) and obtain

$$\begin{aligned}s\big((\bar{v}w)u\big)+s\big(\bar{v}(wu)\big) &= t_B(\bar{v})s(wu)-t_B(w)s(u\bar{v})+t_B(u)s(\bar{v}w)\\ &\quad+t_B(wu)s(\bar{v})-t_B(u\bar{v})s(w)+t_B(\bar{v}w)s(u)\\ &\quad+\lambda\big(n_B(\bar{v}u,w)-t_B(w)n_B(\bar{v},u)\big)\\ &= s(wu)t_B(v)-s(u\bar{v})t_B(w)+s(\bar{v}w)t_B(u)\\ &\quad+s(\bar{v})t_B(wu)-s(w)n_B(u,v)+s(u)n_B(v,w)\\ &\quad+\lambda\big(n_B(u,vw)-t_B(w)n_B(\bar{v},u)\big).\end{aligned}$$

Returning to the definition of α, we conclude

$$\begin{aligned}\alpha = &-s(\bar{w}u)t_B(v)+s(u)n_B(v,w)-s(\bar{w}v)t_B(u)+s(v)n_B(w,u)+s(\bar{w})n_B(v,u)\\ &-\lambda n_B(wv,u)+s(\bar{v}u)t_B(w)-s(wu)t_B(v)+s(u\bar{v})t_B(w)-s(\bar{v}w)t_B(u)\\ &-s(\bar{v})t_B(wu)+s(w)n_B(u,v)-s(u)n_B(v,w)\\ &-\lambda n_B(u,vw)+\lambda t_B(w)n_B(\bar{v},u)\\ = &-\big(s(\bar{w}u)t_B(v)+s(wu)t_B(v)+\lambda t_B(v)t_B(wu)\big)+s(v)\big(t_B(wu)+n_B(w,u)\big)\\ &+\big(s(u)n_B(v,w)-s(u)n_B(v,w)\big)-\big(s(\bar{w}v)+s(\bar{v}w)\big)t_B(u)\\ &+\big(s(\bar{w})+s(w)\big)n_B(u,v)-\lambda n_B(u,vw+wv)\\ &+\big(s(\bar{v}u)+s(u\bar{v})+\lambda n_B(u,\bar{v})\big)t_B(w)\\ = &-s(u)t_B(v)t_B(w)-\lambda t_B(v)t_B(wu)+s(v)t_B(w)t_B(u)-\lambda t_B(u)n_B(v,w)\\ &+\lambda t_B(w)n_B(u,v)-\lambda t_B(v)n_B(w,u)-\lambda t_B(w)n_B(u,v)+\lambda t_B(u)n_B(v,w)\\ &+s(u)t_B(v)t_B(w)+s(\bar{v})t_B(w)t_B(u)-\lambda n_B(u,\bar{v})t_B(w)+\lambda n_B(u,\bar{v})t_B(w)\\ = &-\lambda t_B(v)\big(t_B(wu)+n_B(w,u)\big)+\big(s(v)+s(\bar{v})\big)t_B(w)t_B(u)\\ = &-\lambda t_B(v)t_B(w)t_B(u)+\lambda t_B(v)t_B(w)t_B(u)=0,\end{aligned}$$

and the proof of (4) is complete.

It remains to deal with (5): by (3.2.5), (3.3.3), we have

$$\begin{aligned}n_C\big((vj)(wj),vj\big) &= n_C\Big(\big[-\lambda s(\bar{w}v)1_B+\lambda s(v)w+\lambda s(\bar{w})v-\lambda^2 wv+\mu\bar{w}v\big]\\ &\qquad+\big[s(\bar{w}v)1_B-s(v)w+\lambda wv\big]j,vj\Big)\\ &= -\lambda s(\bar{w}v)s(\bar{v})+\lambda s(v)s(\bar{v}w)+\lambda s(\bar{w})s(\bar{v}v)-\lambda^2 s\big(\bar{v}(wv)\big)\\ &\quad+\mu s\big(\bar{v}(\bar{w}v)\big)-\mu s(\bar{w}v)t_B(v)+\mu s(v)n_B(w,v)-\lambda\mu n_B(wv,v)\end{aligned}$$

Here $\lambda s(\bar{w}v)s(\bar{v})=\lambda^2 s(\bar{w}v)t_B(v)-\lambda s(\bar{w}v)s(v)$, $\lambda s(\bar{w})s(\bar{v}v)=\lambda^2 s(\bar{w})n_B(v)$. Moreover, since s is alternative, Prop. 4.8 implies

$$s\big(\bar{v}(wv)\big)=t_B(v)s(wv)-s(vwv)=s(wv)t_B(v)-n_B(v,\bar{w})s(v)+n_B(v)s(\bar{w})$$

and, similarly,

$$s\big(\bar{v}(\bar{w}v)\big)=s(\bar{w}v)t_B(v)-n_B(v,w)s(v)+n_B(v)s(w).$$

Therefore

$$\begin{aligned} n_C\big((vj)(wj), vj\big) = & -\lambda^2 s(\bar{w}v)t_B(v) + \lambda s(\bar{w}v)s(v) + \lambda s(\bar{v}w)s(v) + \lambda^2 s(\bar{w})n_B(v) \\ & -\lambda^2 s(wv)t_B(v) + \lambda^2 n_B(v,\bar{w})s(v) - \lambda^2 n_B(v)s(\bar{w}) + \mu s(\bar{w}v)t_B(v) \\ & -\mu n_B(v,w)s(v) + \mu n_B(v)s(w) - \mu s(\bar{w}v)t_B(v) + \mu s(v)n_B(v,w) \\ & -\lambda\mu n_B(wv,v) \\ = & -\lambda^2 s(v)t_B(v)t_B(w) + \lambda^2 s(v)n_B(v,w) + \lambda^2 s(v)n_B(v,\bar{w}) + \\ & +\mu\big(s(w) - \lambda t_B(w)\big)n_B(v) \\ = & -\mu s(\bar{w})n_B(v) = n_C(vj)t_C(wj). \end{aligned}$$

Thus (5) holds and the theorem is proved. $\square$

4.11. **Killing the annihilator.** Let B be a conic k-algebra and $\mathfrak{a} := \mathrm{Ann}(B)$. We put $k^\dagger := k/\mathfrak{a}$, write

$$\pi\colon\ k \longrightarrow k^\dagger, \quad \alpha \longmapsto \alpha^\dagger,$$

for the natural projection and, following [14, 2.9], have the canonical identification

$$B^\dagger := B_{k^\dagger} = B/\mathfrak{a}B = B$$

as conic $k^\dagger$-algebras such that

$$u \otimes \alpha^\dagger = \alpha u, \quad u = u \otimes 1_{k^\dagger} \qquad\qquad (\alpha \in k,\ u \in B).$$

More specifically, the $k^\dagger$-module structure of B is given by

$$\alpha^\dagger u = \alpha u \qquad\qquad (\alpha \in k,\ u \in B)$$

and norm, bilinearized norm, trace of $B^\dagger$ act on B according to the rules

$$\begin{aligned} n_{B^\dagger}(u) &= n_B(u)^\dagger\ , \\ n_{B^\dagger}(u,v) &= n_B(u,v)^\dagger\ , \\ t_{B^\dagger}(u) &= t_B(u)^\dagger \end{aligned}$$

for all $u, v \in B$. In particular, $B = B^\dagger$ as $\mathbb{Z}$-algebras and $\iota_{B^\dagger} = \iota_B$ as $\mathbb{Z}$-linear maps since, for example,

$$\iota_{B^\dagger}(u) = t_{B^\dagger}(u)1_{B^\dagger} - u = t_B(u)^\dagger 1_B - u = t_B(u)1_B - u = \iota_B(u)$$

for all $u \in B$.

Now fix a scalar $\mu \in k$ and a linear form $s\colon\ B \to k$. Then $s^\dagger := \pi \circ s\colon\ B^\dagger = B \to k^\dagger$ is the scalar extension of s from k to $k^\dagger$, and since the non-orthogonal Cayley-Dickson construction obviously is compatible with base change, we conclude

$$\mathrm{Cay}(B;\mu,s) = \mathrm{Cay}(B;\mu,s)^\dagger = \mathrm{Cay}(B^\dagger;\mu^\dagger,s^\dagger) = \mathrm{Cay}(B;\mu^\dagger,s^\dagger)$$

as conic $k^\dagger$-algebras. With an eye on Thm. 4.10 we note that $s^\dagger$ is alternative if and only if, for $u, v, w \in B$, the expression $s([u,v,w])$ belongs to $\mathrm{Ann}(B)$ as soon as two of the three arguments u, v, w coincide.

4.12. **Theorem.** *Let B be a conic k-algebra, $\mu \in k$ and $s\colon\ B \to k$ a linear form. With the notation and conventions of* 4.11, *the following conditions are equivalent.*

(i) $\mathrm{Cay}(B;\mu,s)$ *is flexible.*
(ii) *B is flexible, the conjugation of B is an involution and $s^\dagger$ is alternative.*

Proof. Put $C := \mathrm{Cay}(B;\mu,s)$

(i) $\Rightarrow$ (ii). Since C is flexible, so is B and

$$\big(t_C(xy) - n_C(x,\bar{y})\big)x = \big(n_C(x,xy) - t_C(y)n_C(x)\big)1_C$$

for all $x, y \in C$ [16, Prop. 18.8 (b)]. In particular, for $u_1, u_2, v_1 \in B$ and $x := u_1 + v_1 j$, $y := u_2$, we may apply (4.3.1) to conclude that

$$\begin{aligned}\big(t_C(xy) - n_C(x, \bar{y})\big)x &= \Big(t_B(u_1u_2) - n_B(u_1, \bar{u}_2) \\ &\qquad + t_C\big((v_1j)u_2\big) - n_B(v_1j, \bar{u}_2)\Big)(u_1 + v_1j) \\ &= \big(t_B(u_1u_2) - n_B(u_1, \bar{u}_2)\big)(u_1 + v_1j)\end{aligned}$$

belongs to $k1_B$. Comparing Bj-components, we deduce $t_B(u_1u_2) - n_B(u_1, \bar{u}_2) \in \mathrm{Ann}(B)$. Hence the conjugation of B is an involution [16, Prop. 18.8 (a)], and it remains to show that $s^\dagger$ is alternative. Combining Cor. 4.5 with what has been said in 4.11, we first note that $C^\dagger$ is flexible and its conjugation is an involution, forcing $C^\dagger$ to be norm associative by [16, Cor. 18.13] since $\mathrm{Ann}(C^\dagger) = \mathrm{Ann}(B^\dagger) = \{0\}$. Now Thm. 4.10 shows that $s^\dagger$ is alternative.

(ii) $\Rightarrow$ (i). By hypothesis and 4.11, the $k^\dagger$-algebra $B^\dagger$ is flexible and its conjugation is an involution. Since, therefore, $B^\dagger$ is norm-associative [16, Cor. 18.13], so is $C^\dagger$ by Thm. 4.10 since $s^\dagger$ is alternative by hypothesis. Thus C is flexible. $\square$

5. Commutativity

In this section, we wish to find out under what circumstances the non-orthogonal Cayley-Dickson construction leads to commutative algebras. In the classical orthogonal case, this is known to happen if and only if the conic algebra entering into the construction is commutative itself and has trivial conjugation [16, Thm. 20.13 (a)]. Extending this to the general non-orthogonal case turns out to be easy.

5.1. **Commutator relations.** Let B be a conic k-algebra, $\mu \in k$ a scalar, $s\colon B \to k$ a linear form and $C := \mathrm{Cay}(B; \mu, s) = B \oplus Bj$ the corresponding non-orthogonal Cayley-Dickson construction as in 3.2. With $\lambda := s(1_B)$, we claim

$$[u, vj] = \big(-s(\bar{v}u)1_B + 2s(u)v + s(\bar{v})u - 2\lambda vu\big) + \big(v(u - \bar{u})\big)j, \tag{1}$$

$$\begin{aligned}[v_1j, v_2j] = \big(&\lambda s(\bar{v}_1v_2 - \bar{v}_2v_1)1_B + \lambda s(v_1 - \bar{v}_1)v_2 - \lambda s(v_2 - \bar{v}_2)v_1 \\ &+ \lambda^2[v_1, v_2] + \mu(\bar{v}_2v_1 - \bar{v}_1v_2)\big) + \big(s(\bar{v}_2v_1 - \bar{v}_1v_2)1_B \\ &+ s(v_2)v_1 - s(v_1)v_2 - \lambda[v_1, v_2]\big)j\end{aligned} \tag{2}$$

for all $u, v, v_1, v_2 \in B$. Indeed, a straightforward verification based on (3.2.4), (3.2.3) implies

$$\begin{aligned}[u, vj] &= u(vj) - (vj)u \\ &= -s(\bar{v}u)1_B + s(u)v + s(\bar{v})u - \lambda vu + (vu)j + s(u)v - \lambda vu - (v\bar{u})j \\ &= -s(\bar{v}u)1_B + 2s(u)v + s(\bar{v})u - 2\lambda vu + \big(v(u - \bar{u})\big)j,\end{aligned}$$

hence (1), while (3.2.5) yields

$$\begin{aligned}[v_1j, v_2j] &= (v_1j)(v_2j) - (v_2j)(v_1j) \\ &= -\lambda s(\bar{v}_2v_1)1_B + \lambda s(v_1)v_2 + \lambda s(\bar{v}_2)v_1 - \lambda^2v_2v_1 + \mu\bar{v}_2v_1 \\ &\quad + \big(s(\bar{v}_2v_1)1_B - s(v_1)v_2 + \lambda v_2v_1\big)j + \lambda s(\bar{v}_1v_2)1_B - \lambda s(v_2)v_1 \\ &\quad - \lambda s(\bar{v}_1)v_2 + \lambda^2v_1v_2 - \mu\bar{v}_1v_2 - \big(s(\bar{v}_1v_2)1_B - s(v_2)v_1 + \lambda v_1v_2\big)j \\ &= \lambda s(\bar{v}_1v_2 - \bar{v}_2v_1)1_B + \lambda s(v_1 - \bar{v}_1)v_2 - \lambda s(v_2 - \bar{v}_2)v_1 \\ &\quad + \lambda^2[v_1, v_2] + \mu(\bar{v}_2v_1 - \bar{v}_1v_2) \\ &\quad + \big(s(\bar{v}_2v_1 - \bar{v}_1v_2)1_B + s(v_2)v_1 - s(v_1)v_2 - \lambda[v_1, v_2]\big)j,\end{aligned}$$

hence (2).

5.2. **Lemma.** *With the notation and assumptions of* 5.1, *assume that* B *is commutative. Then the following conditions are equivalent.*

(i) $s(u)v = s(v)u$ *for all* $u, v \in B$.
(ii) $s(u)1_B = \lambda u$ *for all* $u \in B$.
(iii) $s(u)v = \lambda uv$ *for all* $u, v \in B$.

Proof. (i) $\Rightarrow$ (ii). Setting $v = 1_B$ in (i) gives (ii).

(ii) $\Rightarrow$ (iii). If (ii) holds, then $s(u)v = (s(u)1_B)v = \lambda uv$ for all $u, v \in B$, giving (iii).

(iii) $\Rightarrow$ (i). Since the right-hand side of (iii) is symmetric in u, v, so is the left, whence (i) holds. □

5.3. **Reduction modulo one.** Let B be a conic algebra over k. Following Loos [10, 1.2] with a slightly different notation, we put $\dot{B} := B/k1_B$ as a k-module and denote by $u \mapsto \dot{u}$ the natural map from B to $\dot{B}$. Note that, if B is a finitely generated projective k-module of rank $n+1$, then $\dot{B}$ is is one of rank n since $1_B \in B$ is unimodular [16, 18.6].

5.4. **Proposition.** *Let* B *be a faithful and commutative conic* k*-algebra. Then the assignment* $s \mapsto \lambda := s(1_B)$ *defines a linear isomorphism from the* k*-module of linear forms on* B *satisfying the equivalent conditions* (i)$-$(iii) *of Lemma* 5.2 *onto the annihilator of* $\dot{B}$.

Proof. If $s\colon B \to k$ is a linear form satisfying the equivalent conditions of Lemma 5.2 and $\lambda := s(1_B)$, then condition (ii) of the lemma implies $\lambda\dot{u} = s(u)\dot{1}_B = 0$, so λ annihilates $\dot{B}$. The assignment $s \mapsto \lambda$ clearly defines a linear map, which by (ii) and faithfulness is injective. Conversely, suppose $\lambda \in k$ annihilates $\dot{B}$. Then $\lambda B \subseteq k1_B$, and by faithfulness again, there is a unique map $s\colon B \to k$ satisfying condition (ii) of Lemma 5.2. Clearly, s is linear and $s(1_B) = \lambda$. □

5.5. **Proposition.** *Let* B *be a conic* k*-algebra,* $\mu \in k$ *a scalar and* $s\colon B \to k$ *a linear form. Then the following conditions are equivalent.*

(i) *The non-orthogonal Cayley-Dickson construction* $\mathrm{Cay}(B; \mu, s)$ *is commutative.*
(ii) B *is commutative with trivial conjugation and* $s(u)v = s(v)u$ *for all* $u, v \in B$.

Proof. We put $C := \mathrm{Cay}(B; \mu, s) = B \oplus Bj$ as in 3.2.

(ii) $\Rightarrow$ (i). If (ii) holds, then an inspection of (5.1.2) shows $[v_1 j, v_2 j] = 0$ for all $v_1, v_2 \in B$. Moreover, given $u, v \in B$, not only $v(u - \bar{u}) = 0$ but also, by Lemma 5.2,

$$-s(\bar{v}u)1_B + 2s(u)v + s(\bar{v})u - 2\lambda vu = -\lambda uv + 2\lambda uv + \lambda uv - 2\lambda uv = 0,$$

hence $[u, vj] = 0$ by (5.1.1). Hence C is commutative.

(i) $\Rightarrow$ (ii). Since C is commutative, so is B (as unital subalgebra). Moreover, (5.1.1) for $v = 1_B$ shows $u = \bar{u}$ for all $u \in B$, so the conjugation ι_B is the identity. Combining all this with (5.1.2), we conclude $0 = [v_1 j, v_2 j] = (s(v_2)v_1 - s(v_1)v_2)j$ for all $v_1, v_2 \in B$, and also the final assertion of (ii) follows. □

6. Associativity

In this section, we wish to find necessary and sufficient conditions for the output of a non-orthogonal Cayley-Dickson construction to be an associative algebra. In the classical orthogonal case, this is known to happen if and only if the conic algebra entering into the construction is commutative associative and its conjugation is an involution [16, Thm. 20.13 (b)]. In the general case we will see that a simple additional property of the linear form involved will be enough to guarantee the same conclusion.

6.1. **Nuclei.** Let A be a non-associative k-algebra. Beside the ordinary nucleus [16, 6.6.6] it is sometimes useful to consider one-sided nuclei that are respectively defined by

$$\mathrm{Nuc_l}(A) := \big\{x \in A \mid [x, A, A] = \{0\}\big\} \qquad \text{(left nucleus)}, \tag{1}$$

$$\mathrm{Nuc_m}(A) := \big\{x \in A \mid [A, x, A] = \{0\}\big\} \qquad \text{(middle nucleus)}, \tag{2}$$

$$\mathrm{Nuc_r}(A) := \big\{x \in A \mid [A, A, x] = \{0\}\big\} \qquad \text{(right nucleus)}. \tag{3}$$

They are obviously k-submodules of A. But using the *associator identity*

$$[xy, z, w] - [x, yz, w] + [x, y, zw] = x[y, z, w] + [x, y, z]w, \tag{4}$$

valid in arbitrary non-associative algebras [16, (8.5.2)], it follows immediately that they are, in fact, subalgebras of A.

There are slight modifications of the preceding concepts depending on the choice of a subalgebra $B \subseteq A$. We define

$$(\mathrm{Nuc_l})_B(A) := \big\{x \in A \mid [x, B, B] = \{0\}\big\}, \tag{5}$$

$$(\mathrm{Nuc_m})_B(A) := \big\{x \in A \mid [B, x, B] = \{0\}\big\}, \tag{6}$$

$$(\mathrm{Nuc_r})_B(A) := \big\{x \in A \mid [B, B, x] = \{0\}\big\}, \tag{7}$$

which are submodules, but, in general, no longer subalgebras, of A. On the other hand, we have

6.2. **Lemma.** *Let A be a non-associative k-algebra and $B \subseteq A$ an associative subalgebra. Then $(\mathrm{Nuc_l})_B(A)B \subseteq (\mathrm{Nuc_l})_B(A)$.*

Proof. Let $x \in (\mathrm{Nuc_l})_B(A)$ and $y, z, w \in B$. In the associator identity (6.1.4), the second (resp. third) summand on the left vanishes because $B \subseteq A$ is a subalgebra and so yz (resp. zw) belongs to B. On the other hand, the first summand on the right of (6.1.4) vanishes because B is an associative algebra, while the second summand does because x belongs to $(\mathrm{Nuc_l})_B(A)$. Thus $[xy, z, w] = 0$, forcing $xy \in (\mathrm{Nuc_l})_B(A)$. □

6.3. **Returning to the non-orthogonal Cayley-Dickson construction.** We now return to our conic k-algebra B, a scalar $\mu \in k$ and a linear form $s\colon B \to k$ to form the non-orthogonal Cayley-Dickson construction $C := \mathrm{Cay}(B; \mu, s) = B \oplus Bj$ as in 3.2; in particular, we put $\lambda := s(1_B)$.

While the algebra C will in general not be alternative, there are certain "nice" elements that behave as if it were.

6.4. **Lemma.** *With the assumptions and notation of* 6.3, *the relations*

$$ju = \big(-s(u)1_B + \lambda u\big) + \bar{u}j, \tag{1}$$

$$j(vj) = \mu\bar{v} + s(\bar{v})j, \tag{2}$$

$$(vj)j = \mu v + (\lambda v)j \tag{3}$$

hold for all $u, v \in B$.

Proof. All three equations follow from (3.2.4)–(3.2.5) by a straightforward computation:

$$\begin{aligned}
ju = (1_Bj)u &= -s(u)1_B + \lambda u + \bar{u}j,\\
j(vj) = (1_Bj)(vj) &= \big(-\lambda s(\bar{v})1_B + \lambda^2 v + \lambda s(\bar{v})1_B - \lambda^2 v + \mu\bar{v}\big)\\
&\quad + \big(s(\bar{v})1_B - \lambda v + \lambda v\big)j\\
&= \mu\bar{v} + \big(s(\bar{v})1_B\big)j,\\
(vj)j = (vj)(1_Bj) &= \big(-\lambda s(v)1_B\big) + \lambda s(v)1_B + \lambda^2 v - \lambda^2 v + \mu v\big)\\
&\quad + \big(s(v)1_B - s(v)1_B + \lambda v\big)j\\
&= \mu v + (\lambda v)j,
\end{aligned}$$

as claimed. □

6.5. **Remark.** By (3.3.2)–(3.3.4) we have $n_B(j) = -\mu$, $n_B(j,\bar v) = s(\bar v)$, $t_B(j) = \lambda$. Hence (6.4.2) (resp. (6.4.3)) amounts to $j(vj) = n_B(j,\bar v)j - n_B(j)\bar v$ (resp. $(vj)j = -n_B(j)v + t_B(j)vj = v(t_B(j)j - n_B(j)1_B) = vj^2$), in agreement with (2.5.1), the formula for the U-operator in conic alternative algebras (resp. the right alternative law).

6.6. **Theorem.** *With the assumptions and notation of* 6.3, *the following conditions are equivalent.*

(i) *The conic algebra* $C = \mathrm{Cay}(B;\mu,s)$ *is associative.*
(ii) *The conic algebra* B *is commutative associative, its conjugation is an involution and the linear form* s *satisfies the relation*

$$s(uv)1_B = s(u)v + s(v)\bar u - \lambda \bar u v \tag{1}$$

for all $u, v \in B$.

Proof. We may assume that B is associative and, by Thm. 4.12, that its conjugation is an involution, which by Thm. 4.12 again implies that C is flexible since $s^\dagger$ is trivially alternative, so we have

$$[x,y,x] = 0, \quad [x,y,z] = -[z,y,x] \tag{2}$$

for all $x, y, z \in C$

In order to prove the theorem, we will investigate the conditions that are necessary and sufficient for the associators $[a,b,c]$ to vanish identically in $a,b,c \in C$. By trilinearity, we may assume $a,b,c \in B \cup Bj$ and thus are left with the following cases.

Case 1. $a,b,c \in B$. Then $[a,b,c] = 0$ since we have assumed that B is associative.

Case 2. $|\{a,b,c\} \cap Bj| = 1$.

Case 2.1. $a \in Bj$, $b,c \in B$. Then

$$a = vj, \quad b = u_1, \quad c = u_2 \quad \text{for some } u_1,u_2,v \in B.$$

Since $vj \equiv j\bar v \bmod B$ by (6.4.1), we have $[a,b,c] = [vj,u_1,u_2] = [j\bar v,u_1,u_2]$ in view of Case 1, so we have to find necessary and sufficient conditions for $j\bar v$ to belong to $(\mathrm{Nuc_l})_B(A)$. By Lemma 6.2, we may assume $v = 1_B$. Using (6.4.1), (3.2.4), we now compute

$$\begin{aligned}
[a,b,c] &= [j,u_1,u_2] = (ju_1)u_2 - j(u_1u_2),\\
(ju_1)u_2 &= \big(\big[-s(u_1)1_B + \lambda u_1\big] + \bar u_1 j\big)u_2 = \big[-s(u_1)u_2 + \lambda u_1u_2\big] + (\bar u_1 j)u_2\\
&= \big[-s(u_1)u_2 + \lambda u_1u_2 - s(u_2)\bar u_1 + \lambda \bar u_1 u_2\big] + [\bar u_1 \bar u_2]j,\\
j(u_1u_2) &= \big[-s(u_1u_2)1_B + \lambda u_1u_2\big] + \overline{u_1u_2}\,j = \big[-s(u_1u_2)1_B + \lambda u_1u_2\big] + [\bar u_2 \bar u_1]j.
\end{aligned}$$

Comparing we see that $[a,b,c] = 0$ for all possible choices of Case 2.1 if and only if B is commutative and $s(u_1u_2)1_B = s(u_1)u_2 + s(u_2)\bar u_1 - \lambda \bar u_1 u_2$ for all $u_1,u_2 \in B$, equivalently, B is commutative associative and (1) holds.

In particular, we have established the implication (i) $\Rightarrow$ (ii) of the theorem, and it remains to establish the implication (ii) $\Rightarrow$ (i). For the remainder of the proof, we therefore assume that B is commutative associative and (1) holds. We must show that C is associative, i.e., $[a,b,c] = 0$ for all $a,b,c \in B \cup Bj$.

By flexibility (2), the discussion of Case 2 will be complete once we have dealt with

Case 2.2. $b \in Bj$, $a,c \in B$. Then

$$a = u_1, \quad b = vj, \quad c = u_2 \quad \text{for some } u_1,u_2,v \in B.$$

We now combine the associator identity (6.1.4) with (6.4.1) and (2) to conclude that $[a,b,c] = [u_1,vj,u_2] = [x,yz,w]$ with $x = u_1$, $y = v$, $z = j$, $w = u_2$ is a $\mathbb{Z}$-linear combination of

$$\begin{aligned}
&[xy,z,w] = [u_1v,j,u_2] && \text{(Case 2.2 with } v = 1_B\text{)},\\
&[x,y,zw] = [u_1,v,ju_2] = -[\bar{u}_2j,v,u_1] && \text{(Case 2.1)},\\
&x[y,z,w] = u_1[v,j,u_2] && \text{(Case 2.2 with } v = 1_B\text{)},\\
&[x,y,z]w = [u_1,v,j]u_2 = -[j,v,u_1]u_2 && \text{(Case 1)}.
\end{aligned}$$

Hence we may assume $v = 1_B$. After this reduction we use (3.2.4), (6.4.1) to compute

$$\begin{aligned}
[a,b,c] &= [u_1,j,u_2] = (u_1j)u_2 - u_1(ju_2),\\
(u_1j)u_2 &= \big[-s(u_2)u_1 + \lambda u_1u_2\big] + [u_1\bar{u}_2]j,\\
u_1(ju_2) &= u_1\big(\big[-s(u_2)1_B + \lambda u_2\big] + \bar{u}_2j\big) = \big[-s(u_2)u_1 + \lambda u_1u_2\big] + u_1(\bar{u}_2j)\\
&= \big[-s(u_2)u_1 + \lambda u_1u_2\big] + \big[-s(u_2u_1)1_B + s(u_1)\bar{u}_2\\
&\quad + s(u_2)u_1 - \lambda\bar{u}_2u_1\big] + [\bar{u}_2u_1]j\\
&= \big[-s(u_2u_1)1_B + s(u_1)\bar{u}_2 - \lambda\bar{u}_2u_1 + \lambda u_1u_2\big] + [u_1\bar{u}_2]j.
\end{aligned}$$

Comparing the final expressions of the last two equations by means of (1), we see that they are the same, forcing $[a,b,c] = 0$, as desired.

Case 3. $|\{a,b,c\} \cap Bj| = 2$.

Case 3.1. $a,b \in Bj$, $c \in B$. Then

$$a = v_1j, \quad b = v_2j, \quad c = u \quad \text{for some } u,v_1,v_2 \in B.$$

This time combining the associator identity (6.1.4) with (2), (6.4.2) we conclude that $[a,b,c] = [v_1j,v_2j,u] = [xy,z,w]$ with $x = v_1$, $y = j$, $z = v_2j$, $w = u$ is a $\mathbb{Z}$-linear combination of

$$\begin{aligned}
&x[y,z,w] = v_1[j,v_2j,u] && \text{(Case 3.1 with } v_1 = 1_B\text{)},\\
&[x,y,z]w = [v_1,j,v_2j]u = -[v_2j,j,v_1]u && \text{(Case 3.1 with } v_2 = 1_B\text{)},\\
&[x,yz,w] = [v_1,jv_2j,u] = s(\bar{v}_2)[v_1,j,u] && \text{(Case 2.2)},\\
&[x,y,zw] = [v_1,j,zw] = -[zw,j,v_1] && \text{(Case 2.2, and Case 3.1 with } v_2 = 1_B\text{)}.
\end{aligned}$$

Hence we may assume $v_1 = 1_B$ or $v_2 = 1_B$.

Case 3.1.1. $v_1 = 1_B$. Then

$$a = j, \quad b = vj, \quad c = u \quad \text{for some } u,v \in B.$$

Using (6.4.2), (6.4.1), (3.2.4) we compute

$$\begin{aligned}
[a,b,c] &= [j,vj,u] = (jvj)u - j\big((vj)u\big),\\
(jvj)u &= \big(\mu\bar{v} + s(\bar{v})j\big)u = \mu\bar{v}u + s(\bar{v})(ju)\\
&= \mu\bar{v}u + s(\bar{v})\big(\big[-s(u)1_B + \lambda u\big] + \bar{u}j\big)\\
&= \big[-s(u)s(\bar{v})1_B + \lambda s(\bar{v})u + \mu u\bar{v}\big] + \big[s(\bar{v})\bar{u}\big]j,\\
j\big((vj)u\big) &= j\big(\big[-s(u)v + \lambda vu\big] + [v\bar{u}]j\big)\\
&= -s(u)jv + \lambda j(vu) + j(v\bar{u})j\\
&= \big[s(u)s(v)1_B - \lambda s(u)v\big] - \big[s(u)\bar{v}\big]j\\
&\quad + \big[-\lambda s(vu)1_B + \lambda^2vu\big] + [\lambda\bar{u}\bar{v}]j + \mu u\bar{v} + s(u\bar{v})j\\
&= \big[\big(s(u)s(v) - \lambda s(uv)\big)1_B - \lambda s(u)v + \lambda^2uv + \mu u\bar{v}\big]\\
&\quad + \big[s(u\bar{v})1_B - s(u)\bar{v} + \lambda\bar{u}\bar{v}\big]j.
\end{aligned}$$

Comparing by means of (1), we obtain

$$\begin{aligned}[a,b,c] &= \big[\lambda s(uv)1_B - s(u)s(v)1_B - s(u)s(\bar v)1_B + \lambda s(u)v + \lambda s(\bar v)u - \lambda^2 uv\big] \\ &\quad - \big[s(u\bar v)1_B - s(u)\bar v - s(\bar v)\bar u + \lambda \bar u\bar v\big]j \\ &= \lambda s(uv)1_B - \lambda t_B(v)s(u)1_B + \lambda s(u)v + \lambda s(\bar v)u - \lambda^2 uv \\ &= \lambda s(uv)1_B - \lambda s(u)\bar v + \lambda^2 t_B(v)u - \lambda s(v)u - \lambda^2 uv \\ &= \lambda\big(s(vu)1_B - s(v)u - s(u)\bar v + \lambda \bar v u\big) = 0,\end{aligned}$$

as desired.

Case 3.1.2. $v_2 = 1_B$. Then

$$a = vj, \quad b = j, \quad c = u \quad \text{for some } u, v \in B.$$

Using (6.4.3), (3.2.4), (6.4.1), we compute

$$\begin{aligned}[a,b,c] &= \big((vj)j\big)u - (vj)(ju), \\ \big((vj)j\big)u &= \big(\mu v + (\lambda v)j\big)u = \mu vu + \lambda(vj)u \\ &= \mu uv + \big[-\lambda s(u)v + \lambda^2 vu\big] + \lambda(v\bar u)j \\ &= \big[-\lambda s(u)v + (\lambda^2+\mu)uv\big] + [\lambda \bar u v]j, \\ (vj)(ju) &= (vj)\big(\big[-s(u)1_B + \lambda u\big] + \bar u j\big) \\ &= -s(u)vj + \lambda(vj)u + (vj)(\bar u j) \\ &= \big[-s(u)v\big]j + \big[-\lambda s(u)v + \lambda^2 vu\big] + [\lambda v\bar u]j \\ &\quad + \big[-\lambda s(uv)1_B + \lambda s(v)\bar u + \lambda s(u)v - \lambda^2 \bar u v + \mu uv\big] \\ &\quad + \big[s(uv)1_B - s(v)\bar u + \lambda \bar u v\big]j \\ &= \big[-\lambda s(uv)1_B + \lambda s(v)\bar u + \lambda^2(u-\bar u)v + \mu uv\big] \\ &\quad + \big[s(uv)1_B - s(u)v - s(v)\bar u + 2\lambda \bar u v\big]j\end{aligned}$$

Comparing and using (1), we conclude

$$\begin{aligned}[a,b,c] &= \big[\lambda s(uv)1_B - \lambda s(u)v - \lambda s(v)\bar u + \lambda^2 uv - \lambda^2(u-\bar u)v\big] \\ &\quad + \big[\lambda \bar u v - s(uv)1_B + s(u)v + s(v)\bar u - 2\lambda \bar u v\big]j \\ &= \lambda\big[s(uv)1_B - s(u)v - s(v)\bar u + \lambda \bar u v\big] \\ &\quad - \big[s(uv)1_B - s(u)v - s(v)\bar u + \lambda \bar u v\big]j = 0,\end{aligned}$$

as desired.

Case 3.2. $a, c \in Bj$, $b \in B$. Then

$$a = v_1 j, \quad b = u, \quad c = v_2 j \quad \text{for some } u, v_1, v_2 \in B.$$

Combining the associator identity (6.1.4) with (2), we conclude that

$$[a,b,c] = [v_1 j, u, v_2 j] = [xy, z, w]$$

with $x = v_1$, $y = j$, $z = u$, $w = v_2 j$ is a $\mathbb{Z}$-linear combination of

$x[y,z,w] = v_1[j,u,v_2 j] = -v_1[v_2 j, u, j]$	(Case 3.2 with $v_2 = 1_B$),
$[x,y,z]w = [v_1, j, u]w$	(Case 2.2),
$[x,yz,w] = [v_1, ju, v_2 j]$	(Cases 2.1, 3.1 because of (6.4.1) and (2)),
$[x,y,zw] = [v_1, j, zw]$	(Cases 2.2, 3.1 because of (2)).

We may thus assume $v_2 = 1_B$. Then

$$a = vj, \quad b = u, \quad c = j \quad \text{for some } u, v \in B.$$

Using (3.2.4), (6.4.3), (3.2.5) we compute

$$
\begin{aligned}
[a,b,c] &= [vj,u,j] = \big((vj)u\big)j - (vj)(uj), \\
\big((vj)u\big)j &= \big(\big[-s(u)v+\lambda vu\big]+[v\bar{u}]j\big)j = \big[-s(u)v+\lambda uv\big]j + \big([v\bar{u}]j\big)j \\
&= \big[-s(u)v+\lambda uv\big]j + \mu\bar{u}v + [\lambda\bar{u}v]j \\
&= \mu\bar{u}v + \big[-s(u)v+\lambda t_B(u)v\big]j \\
&= [\mu\bar{u}v] + [s(\bar{u})v]j, \\
(vj)(uj) &= \big[-\lambda s(\bar{u}v)1_B + \lambda s(v)u + \lambda s(\bar{u})v - \lambda^2 uv + \mu\bar{u}v\big] \\
&\quad \big[s(\bar{u}v)1_B - s(v)u + \lambda uv\big]j,
\end{aligned}
$$

which by (1) implies

$$
\begin{aligned}
[a,b,c] &= \lambda\big[s(\bar{u}v)1_B - s(\bar{u})v - s(v)u + \lambda uv\big] - \big[s(\bar{u}v)1_B - s(\bar{u})v - s(v)u + \lambda uv\big]j \\
&= 0,
\end{aligned}
$$

as desired.

Case 4. $a,b,c \in Bj$. Then

$$
a = v_1j, \quad b = v_2j, \quad c = v_3j \quad \text{for some } v_1,v_2,v_3 \in B.
$$

Again we make use of the associator identity (6.1.4) combined with (2), (6.4.2) to conclude that $[a,b,c] = [v_1j,v_2j,v_3j] = [x,yz,w]$ with $x = v_1j$, $y = v_2$, $z = j$, $w = v_3j$ is a $\mathbb{Z}$-linear combination of

$$
\begin{aligned}
&x[y,z,w] = (v_1j)[v_2,j,v_3j] = -(v_1j)[v_3j,j,v_2] && \text{(Case 3.1)}, \\
&[x,y,z]w = [v_1j,v_2,j]w && \text{(Case 3.2)}, \\
&[xy,z,w] = [(v_1j)v_2,j,v_3j] = -[v_3j,j,(v_1j)v_2] && \text{(Case 3.1, Case 4 for } v_2 = 1_B), \\
&[x,y,zw] = [v_1j,v_2,jv_3j] && \text{(Cases 2.1, 3.2)}.
\end{aligned}
$$

We are thus reduced to the case $v_2 = 1_B$ and then have

$$
a = vj, \quad b = j, \quad c = wj \quad \text{for some } v,w \in B.
$$

Using (6.4.3), (3.2.3), (6.4.2), (3.2.4), we compute

$$
\begin{aligned}
[a,b,c] &= [vj,j,wj] = \big((vj)j\big)(wj) - (vj)(jwj), \\
\big((vj)j\big)(wj) &= \big(\mu v + [\lambda v]j\big)(wj) = \mu v(wj) + \lambda(vj)(wj) \\
&= \big[-\mu s(\bar{w}v)1_B + \mu s(v)w + \mu s(\bar{w})v - \lambda\mu wv\big] + [\mu wv]j \\
&\quad + \big[-\lambda^2 s(\bar{w}v)1_B + \lambda^2 s(v)w + \lambda^2 s(\bar{w})v - \lambda^3 wv + \lambda\mu\bar{w}v\big] \\
&\quad + \big[\lambda s(\bar{w}v)1_B - \lambda s(v)w + \lambda^2 wv\big]j \\
&= \big[-(\lambda^2+\mu)\big(s(\bar{w}v)1_B - s(\bar{w})v - s(v)w + \lambda wv\big) + \lambda\mu v\bar{w}\big] \\
&\quad + \big[\mu vw + \lambda\big(s(\bar{w}v)1_B - s(v)w + \lambda wv\big)\big]j \\
&= [\lambda\mu v\bar{w}] + \big[\lambda s(\bar{w})v + \mu vw\big]j, \\
(vj)(jwj) &= (vj)\big(\mu\bar{w} + s(\bar{w})j\big) = \mu(vj)\bar{w} + s(\bar{w})(vj)j \\
&= \big[-\mu s(\bar{w})v + \lambda\mu v\bar{w}\big] + [\mu vw]j + \big[\mu s(\bar{w})v\big] + \big[\lambda s(\bar{w})v\big]j \\
&= [\lambda\mu v\bar{w}] + \big[\lambda s(\bar{w})v + \mu vw\big]j
\end{aligned}
$$

Comparing, we conclude $[a,b,c] = 0$, which completes the proof of the theorem. $\square$

6.7. **Reminder: quadratic algebras.** Following Knus [7, (1.3.6)], a k-algebra R is said to be *quadratic* if it contains a unit element and is finitely generated projective of rank 2 as a k-module. In this case, R is a conic algebra, with norm, trace respectively given by $n_R(x) = \det(L_x)$, $t_R(x) = \mathrm{tr}(L_x)$ for all $x \in R$.

6.8. **Corollary.** *Let R be a quadratic k-algebra, $\mu \in k$ a scalar and $s\colon R \to k$ a linear form. Then the conic algebra* $\mathrm{Cay}(R;\mu,s)$ *is associative.*

Proof. Since R is commutative associative and its conjugation is an involution,, it will be enough by Thm. 6.6 to show that s satisfies equation (6.6.1). Localizing if necessary, we may assume that R is a free k-module of rank 2, with basis $1_C, w$, for some $w \in R$. Thanks to bilinearity, it suffices to establish (6.6.1) for $u = 1_C$, $v = 1_C$ and $u = v = w$. Indeed,

$$\begin{aligned} s(1_R)v + s(v)\bar{1}_R - \lambda\bar{1}_R v &= \lambda v + s(v)1_R - \lambda v = s(1_R v)1_R, \\ s(u)1_R + s(1_R)\bar{u} - \lambda\bar{u}1_R &= s(u)1_R + \lambda\bar{u} - \lambda\bar{u} = s(u1_R)1_R, \\ s(w)w + s(w)\bar{w} - \lambda\bar{w}w &= t_R(w)s(w)1_R - \lambda n_R(w)1_R \\ &= s\big(t_R(w)w - n_R(w)1_R\big)1_R = s(w^2)1_R, \end{aligned}$$

as claimed. □

6.9. **Corollary.** *Let B be a commutative associative conic k-algebra whose conjugation is an involution and $\mu \in k$ a scalar. Then the conic algebra* $\mathrm{Cay}(B;\mu,t_B)$ *is associative.*

Proof. By Thm. 6.6, it suffices to show that $s := t_B$ satisfies equation (6.6.1). Since $\lambda = s(1_B) = t_B(1_B) = 2$, we have

$$\begin{aligned} t_B(u)v + t_B(v)\bar{u} - 2\bar{u}v &= t_B(u)v + t_B(v)\bar{u} - \bar{u}\circ v \\ &= t_B(u)v + t_B(v)\bar{u} - t_B(\bar{u})v - t_B(v)\bar{u} + n_B(\bar{u},v)1_B \\ &= t_B(u)v + t_B(v)\bar{u} - t_B(u)v - t_B(v)\bar{u} + t_B(uv)1_B \\ &= t_B(uv)1_B, \end{aligned}$$

as claimed. □

6.10. **Corollary.** *Let B be a commutative associative conic k-algebra that is projective of rank at least 3. Suppose further $\mu \in k$ is a scalar and $s\colon B \to k$ is a linear form such that the conic algebra* $\mathrm{Cay}(B;\mu,s)$ *is associative. Then*

$$s(u)t_B(v) = s(v)t_B(u)$$

for all $u, v \in B$. In particular, if (in addition to the above) t_B is surjective, then $s = \alpha t_B$ for some $\alpha \in k$.

Proof. By Thm. 6.6, the linear form s satisfies the equation

$$s(uv)1_B = s(u)v + s(v)\bar{u} - \lambda\bar{u}v \qquad (\lambda := s(1_B)) \tag{1}$$

for all $u, v \in B$. Since the left-hand side is symmetric in u, v, so is the right and we have

$$s(u)v + s(v)\bar{u} - \lambda\bar{u}v = s(v)u + s(u)\bar{v} - \lambda\bar{v}u.$$

(This also follows from applying the conjugation to (1).) We now conclude

$$s(u)(v - \bar{v}) - s(v)(u - \bar{u}) - \lambda(\bar{u}v - u\bar{v}) = 0.$$

But since $\bar{u}v - u\bar{v} = t_B(u)v - uv - t_B(v)u + uv = t_B(u)v - t_B(v)u$, $u - \bar{u} = u - t_B(u)1_B + u = 2u - t_B(u)1_B$ and, similarly, $v - \bar{v} = 2v - t_B(v)1_B$, we obtain

$$s(u)\big(2v - t_B(v)1_B\big) - s(v)\big(2u - t_B(u)1_B\big) - \lambda\big(t_B(u)v - t_B(v)u\big) = 0.$$

Writing the left-hand side as a linear combination of $v, u, 1_B$, we finally end up with

$$\big(2s(u) - \lambda t_B(u)\big)v - \big(2s(v) - \lambda t_B(v)\big)u - \big(s(u)t_B(v) - s(v)t_B(u)\big)1_B = 0. \tag{2}$$

Since the linear form $u \mapsto 2s(u) - \lambda t_B(u)$ obviously kills 1_B, the expression $2s(\dot{u}) - \lambda t_B(\dot{u})$ makes sense for $u \in B$ (although the individual terms $2s(\dot{u})$ and $\lambda t_B(\dot{u})$ do not). Thus, reading (2) in $\dot{B}$, we conclude

$$\big(2s(\dot{u}) - \lambda t_B(\dot{u})\big)\dot{v} = \big(2s(\dot{v}) - \lambda t_B(\dot{v})\big)\dot{u} \qquad (\dot{u}, \dot{v} \in \dot{B}). \tag{3}$$

Localizing if necessary, we may assume that $\dot{B}$ is free of rank at least 2 as a k-module. Picking a basis $(\dot{e}_i)_{i\in I}$, $|I| \geq 2$, of $\dot{B}$, equation (3) implies $2s(\dot{e}_i) - \lambda t_B(\dot{e}_i) = 0 = 2s(\dot{e}_j) - \lambda t_B(\dot{e}_j)$ for all $i, j \in I$, $i \neq j$. Hence $2s(u) = \lambda t_B(u)$ for all $u \in B$, and since 1_B is unimodular [16, 18.6], (2) yields the first assertion of the corollary: $s(u)t_B(v) = s(v)t_B(u)$ for all $u, v \in B$. If t_B is surjective, some $v \in B$ has $t_B(v) = 1$, and the second assertion follows as well. □

7. Alternativity

This section is devoted to the problem of finding conditions that are necessary and sufficient for the output of the non-orthogonal Cayley-Dickson construction to be an alternative conic algebra. In the classical orthogonal case, this is known to happen if and only if the conic algebra entering into the construction is associative and its conjugation is an involution [16, Thm. 20.13 (c)]. In the general case, a certain alternating trilinear map will have to vanish identically in order to reach the same conclusion.

Throughout this section, we fix a conic k-algebra B, a scalar $\mu \in k$ and a linear form $s\colon B \to k$ in order to from the conic algebra $C := \mathrm{Cay}(B; \mu, s) = B \oplus Bj$ as in 3.2. As usual, we put $\lambda := s(1_B)$.

7.1. **Proposition.** *If B is multiplicative alternative, then $H_{B,s}\colon B^3 \to B$ defined by*

(1)
$$\begin{aligned} H_{B,s}(u, v, w) = &-s\big((uv)w\big)1_B + s(uv)\bar{w} + s(vw)\bar{u} - s(u\bar{w})v \\ &+ s(u)vw - s(v)\bar{u}\bar{w} - s(w)\bar{u}v + \lambda(\bar{u}v)\bar{w} \end{aligned}$$

for all $u, v, w \in B$ is an alternating trilinear map such that

(2)
$$H_{B,s}(u, v, 1_B) = H_{B,s}(u, v, uv) = 0$$

for all $u, v \in B$.

Proof. The map $H_{B,s}$ is clearly trilinear, so it remains to show that it is alternating and satisfies (2). Noting that the conjugation of B by [16, Prop. 19.2] is an involution, we begin with the former by abbreviating $H := H_{B,s}$, using (2.2.1) and computing

$$\begin{aligned} H(u, u, w) = &-t_B(u)s(uw)1_B + n_B(u)s(w)1_B + t_B(u)s(u)\bar{w} - \lambda n_B(u)\bar{w} + s(uw)\bar{u} \\ &- s(u\bar{w})u + s(u)uw - s(u)\bar{u}\bar{w} - n_B(u)s(w)1_B + \lambda n_B(u)\bar{w}. \end{aligned}$$

By (2.2.4), we have

$$-t_B(u)s(uw)1_B + s(uw)\bar{u} = -s(uw)u, \quad t_B(u)s(u)\bar{w} - s(u)\bar{u}\bar{w} = s(u)u\bar{w}.$$

Hence

$$\begin{aligned} H(u, u, w) &= -s(uw)u + s(u)u\bar{w} - s(u\bar{w})u + s(u)uw \\ &= -t_B(w)s(u)u + t_B(w)s(u)u = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} H(u, v, v) = &-t_B(v)s(uv)1_B + n_B(v)s(u)1_B + s(uv)\bar{v} + t_B(v)s(v)\bar{u} - \lambda n_B(v)\bar{u} \\ &- s(u\bar{v})v + t_B(v)s(u)v - n_B(v)s(u)1_B - s(v)\bar{u}\bar{v} - s(v)\bar{u}v + \lambda n_B(v)\bar{u}. \end{aligned}$$

Since $-t_B(v)s(uv)1_B + s(uv)\bar{v} = -s(uv)v$, $-s(u\bar{v})v + t_B(v)s(u)v = s(uv)v$, $-s(v)\bar{u}\bar{v} - s(v)\bar{u}v = -t_B(v)s(v)\bar{u}$, we have $H(u, v, v) = -s(uv)v + s(uv)v = 0$, and summing up it follows that H is alternating. We now verify (2). First of all,

$$H(u, v, 1_B) = -s(uv)1_B + s(uv)1_B + s(v)\bar{u} - s(u)v + s(u)v - s(v)\bar{u} - \lambda\bar{u}v + \lambda\bar{u}v = 0.$$

Moreover,

$$\begin{aligned} H(u, v, uv) = &-t_B(uv)s(uv)1_B + \lambda n_B(uv)1_B + s(uv)\overline{uv} + s(vuv)\bar{u} - s(u\bar{v}\bar{u})v \\ &+ s(u)vuv - s(v)\bar{u}\bar{v}\bar{u} - s(uv)\bar{u}v + \lambda\bar{u}v\bar{v}\bar{u}. \end{aligned}$$

Here we combine the relation $-t_B(uv)s(uv)1_B + s(uv)\overline{uv} = -s(uv)uv$ with (2.5.1) to obtain

$$\begin{aligned}H(u,v,uv) = &-s(uv)uv + \lambda n_B(uv)1_B + n_B(v,\bar{u})s(v)\bar{u} - n_B(v)s(\bar{u})\bar{u} - t_B(u)s(u\bar{v})v \\ &+ n_B(u,v)s(u)v - n_B(u)s(v)v + n_B(v,\bar{u})s(u)v - n_B(v)s(u)\bar{u} \\ &- n_B(u,\bar{v})s(v)\bar{u} + n_B(u)s(v)v - s(uv)\bar{u}v + \lambda n_B(v)t_B(u)\bar{u} \\ &- \lambda n_B(v)n_B(u)1_B.\end{aligned}$$

Canceling out the second (resp. third, seventh) term against the last (resp. tenth, eleventh) one on the right-hand side and regrouping, we obtain

$$\begin{aligned}H(u,v,uv) = &-s(uv)uv - t_B(u)s(u\bar{v})v - s(uv)\bar{u}v - n_B(v)s(\bar{u})\bar{u} - n_B(v)s(u)\bar{u} \\ &+ \lambda n_B(v)t_B(u)\bar{u} + n_B(u,v)s(u)v + n_B(\bar{u},v)s(u)v \\ = &-t_B(u)t_B(v)s(u)v - \lambda n_B(v)t_B(u)\bar{u} + \lambda n_B(v)t_B(u)\bar{u} + t_B(u)t_B(v)s(u)v,\end{aligned}$$

and this is zero as claimed. ☐

7.2. **Local generation.** Let r be a natural number. A unital non-associative algebra A over k is said to be *locally generated by r elements* if, for every $\mathfrak{p} \in \mathrm{Spec}(k)$, the $k_{\mathfrak{p}}$-algebra $A_{\mathfrak{p}}$ is unitally generated by r elements in the usual sense. For example, a quadratic k-algebra (cf. 6.7) is locally generated by a single element. Now suppose B is a quaternion algebra over k [16, 21.19]. Locally, B arises from a quadratic étale algebra by means of the Cayley-Dickson construction [16, Cor. 21.16]. *Hence quaternion algebras are locally generated by two elements.*

7.3. **Corollary.** *If B as in* 7.1 *is locally generated by two elements, then $H_{B,s} = 0$.*

Proof. The assertion is local on k, so we may assume that k is a local ring. If B is unitally generated by $x, y \in B$, then [16, Exc. 70] shows that it is spanned by $1_B, x, y, xy$ as a k-module. Since $H := H_{B,s}$ is alternating trilinear, the assertion will follow once we have shown $H(u,v,w) = 0$ for *distinct* elements $u, v, w \in \{1_B, x, y, xy\}$. But in view of Prop. 7.1, this is obvious. ☐

7.4. **Theorem.** *If B is associative and its conjugation is an involution, then*

$$[u_1 + v_1 j, u_1 + v_1 j, u_2 + v_2 j] = \overline{H_{B,s}(u_1, \bar{v}_1, u_2)} + H_{B,s}(\bar{u}_1, \bar{v}_1, v_2)j \tag{1}$$

for all $u_1, u_2, v_1, v_2 \in B$.

Proof. We proceed in several steps.

1^0. Since B is associative, every linear form on B as well $B^\dagger$ (cf. 4.11) is trivially alternative (cf. 4.7), and as B is flexible, so is C, by Thm. 4.12. With $H := H_{B,s}$ we now claim that it suffices to show

$$[u,u,w] = 0, \tag{2}$$

$$[vj,u,u] = 0, \tag{3}$$

$$[vj,vj,u] = 0, \tag{4}$$

$$[wj,vj,vj] = 0, \tag{5}$$

$$[u,vj,w] + [vj,u,w] = \overline{H(u,\bar{v},w)}, \tag{6}$$

$$[wj,u,vj] + [wj,vj,u] = -H(\bar{u},\bar{v},w)j \tag{7}$$

for all $u, v, w \in B$. Indeed, if these relations are fulfilled, we expand the left hand side of (1) by using flexibility of C to obtain

$$\begin{aligned}[u_1+v_1j,u_1+v_1j,u_2+v_2j] &= [u_1,u_1,u_2]+[u_1,u_1,v_2j]+[u_1,v_1j,u_2]+[u_1,v_1j,v_2j] \\ &\quad +[v_1j,u_1,u_2]+[v_1j,u_1,v_2j]+[v_1j,v_1j,u_2] \\ &\quad +[v_1j,v_1j,v_2j] \\ &= [u_1,u_1,u_2]-[v_2j,u_1,u_1]+\big([u_1,v_1j,u_2]+[v_1j,u_1,u_2]\big) \\ &\quad -\big([v_2j,u_1,v_1j]+[v_2j,v_1j,u_1]\big)+[v_1j,v_1j,u_2] \\ &\quad -[v_2j,v_1j,v_1j],\end{aligned}$$

which by (2)−(7) collapses to

$$[u_1+v_1j,u_1+v_1j,u_2+v_2j] = \overline{H(u_1,\bar v_1,u_2)}+H(\bar u_1,\bar v_1,v_2)j,$$

as claimed.

2^0. We now turn to (2), which is obvious since B is associative by hypothesis.

3^0. In order to prove (3), we start out from $[vj,u,u]=((vj)u)u-(vj)u^2$, where (3.2.3), (2.2.6) give

$$\begin{aligned}\big((vj)u\big)u &= \Big(\big[-s(u)v+\lambda vu\big]+\big[v\bar u\big]j\Big)u = \big[-s(u)vu+\lambda vu^2\big]+\big([v\bar u]j\big)u \\ &= \big[-s(u)vu+\lambda vu^2-s(u)v\bar u+\lambda v\bar u u\big]+\big[v\bar u^2\big]j \\ &= \big[-t_B(u)s(u)v+\lambda vu^2+\lambda n_B(u)v\big]+\big[v\bar u^2\big]j, \\ (vj)u^2 &= \big[-s(u^2)v+\lambda vu^2\big]+\big[v\bar u^2\big]j \\ &= \big[-t_B(u)s(u)v+\lambda n_B(u)v+\lambda vu^2\big]+\big[v\bar u^2\big]j,\end{aligned}$$

and subtracting the second expression from the first yields (3).

4^0. Equation (4) is a bit more troublesome. We have $[vj,vj,u]=(vj)^2u-(vj)((vj)u)$, where we treat the terms on the right separately. First of all, since C is a conic algebra, with norm and trace given as in Prop. 3.3, we obtain $(vj)^2=t_C(vj)(vj)-n_C(vj)1_C$, hence

$$(vj)^2=\big[\mu n_B(v)1_B\big]+\big[s(\bar v)v\big]j. \tag{8}$$

This implies $(vj)^2u=\big[\mu n_B(v)u\big]+\big(\big[s(\bar v)v\big]j\big)u$, and from (3.2.3)we conclude

$$(vj)^2u=\big[\mu n_B(v)u-s(u)s(\bar v)v+\lambda s(\bar v)vu\big]+\big[s(\bar v)v\bar u\big]j. \tag{9}$$

On the other hand, by (3.2.3) again and (3.2.5),

$$\begin{aligned}(vj)\big((vj)u\big) &= (vj)\Big(\big[-s(u)v+\lambda vu\big]+\big[v\bar u\big]j\Big) \\ &= \big[s(u)s(v)v-\lambda s(vu)v-\lambda s(u)v^2+\lambda^2v^2u\big]+\big[-s(u)v\bar v+\lambda v\bar u\bar v\big]j \\ &\quad +\big[-\lambda s(u\bar v v)1_B+\lambda s(v)v\bar u+\lambda s(u\bar v)v-\lambda^2v\bar u v+\mu u\bar v v\big] \\ &\quad +\big[s(u\bar v v)1_B-s(v)v\bar u+\lambda v\bar u v\big]j\end{aligned}$$

Here we observe $\lambda v\bar u\bar v+\lambda v\bar u v=\lambda t_B(v)v\bar u$, which simplifies the coefficient of j and together with (2.2.1), (2.2.6), (2.5.1) implies

$$\begin{aligned}(vj)\big((vj)u\big) &= \big[s(u)s(v)v-\lambda s(vu)v-\lambda t_B(v)s(u)v \\ &\quad +\lambda n_B(v)s(u)1_B+\lambda^2t_B(v)vu-\lambda^2n_B(v)u\big] \\ &\quad +\big[-n_B(v)s(u)1_B+\lambda t_B(v)v\bar u\big]j \\ &\quad +\big[-\lambda n_B(v)s(u)1_B+\lambda s(v)v\bar u+\lambda s(u\bar v)v-\lambda^2n_B(v,u)v \\ &\quad +\lambda^2n_B(v)u+\mu n_B(v)u\big]+\big[n_B(v)s(u)1_B-s(v)v\bar u\big]j.\end{aligned}$$

In the B-component (resp. the Bj-component), the term(s) $\lambda n_B(v)s(u)1_B$ and $\lambda^2 n_B(v)u$ (resp. $n_B(v)s(u)1_B$) cancel, while we have $\lambda s(u\bar{v})v - \lambda t_B(v)s(u)v = -\lambda s(uv)v$ and $\lambda t_B(v)v\bar{u} - s(v)v\bar{u} = s(\bar{v})v\bar{u}$. Combining this with (2.2.4), (2.2.5), we conclude

$$\begin{aligned}
(vj)\big((vj)u\big) &= \big[s(u)s(v)v - \lambda s(vu)v - \lambda s(uv)v + \lambda^2 t_B(v)vu + \lambda t_B(u)s(v)v \\
&\quad - \lambda s(v)vu - \lambda^2 n_B(u,v)v + \mu n_B(v)u\big] + \big[s(\bar{v})v\bar{u}\big]j \\
&= \big[s(u)s(v)v - \lambda t_B(u)s(v)v - \lambda t_B(v)s(u)v + \lambda^2 n_B(u,v)v \\
&\quad + \lambda^2 t_B(v)vu + \lambda t_B(u)s(v)v - \lambda s(v)vu - \lambda^2 n_B(u,v)v + \mu n_B(v)u\big] \\
&\quad + \big[s(\bar{v})v\bar{u}\big]j \\
&= \big[\mu n_B(v)u - s(u)s(\bar{v})v + \lambda s(\bar{v})vu\big] + \big[s(\bar{v})v\bar{u}\big]j,
\end{aligned}$$

and by (9) this agrees with $(vj)^2 u$.

5^0. Passing to (5), we begin by noting that Lemma 6.4 implies $wj \equiv j\bar{w} \bmod B$. Since $[B, vj, vj] = [vj, vj, B] = \{0\}$ by (4) and flexibility, it therefore suffices to show $[jw, vj, vj] = 0$, equivalently, after an obvious change of notation,

$$\big((jv)(wj)\big)(wj) = (jv)(wj)^2. \tag{10}$$

In order to prove this, we first show

$$(jv)(wj) = \big[\mu\overline{vw}\big] + \big[s(\overline{vw})1_B\big]j, \tag{11}$$

which is just the middle Moufang identity in disguise (but still requires a proof since C need not be alternative). Indeed, from (6.4.1), (3.2.4), (3.2.5) we deduce

$$\begin{aligned}
(jv)(wj) &= \Big(\big[-s(v)1_B + \lambda v\big] + \bar{v}j\Big)(wj) \\
&= \big[-s(v)w\big]j + \big[-\lambda s(\bar{w}v)1_B + \lambda s(v)w + \lambda s(\bar{w})v - \lambda^2 wv\big] + \big[\lambda wv\big]j \\
&\quad + \big[-\lambda s(\bar{w}\bar{v})1_B + \lambda s(\bar{v})w + \lambda s(\bar{w})\bar{v} - \lambda^2 w\bar{v} + \mu\bar{w}\bar{v}\big] \\
&\quad + \big[s(\bar{w}\bar{v})1_B - s(\bar{v})w + \lambda w\bar{v}\big]j \\
&= \big[-\lambda t_B(v)s(\bar{w})1_B + \lambda^2 t_B(v)w + \lambda t_B(v)s(\bar{w})1_B - \lambda^2 t_B(v)w + \mu\overline{vw}\big] \\
&\quad + \big[-\lambda t_B(v)w + \lambda t_B(v)w + s(\overline{vw})1_B\big]j,
\end{aligned}$$

and (11) follows. We are now ready to tackle (10) and first deal with the right-hand side. Since C is a conic algebra with norm, trace given as in Prop. 3.3, we combine (6.4.1) with (11) and obtain

$$\begin{aligned}
(jv)(wj)^2 &= t_B(wj)(jv)(wj) - n_B(wj)(jv) = \mu n_B(w)(jv) + s(\bar{w})(jv)(wj) \\
&= \big[-\mu n_B(w)s(v)1_B + \lambda\mu n_B(w)v\big] + \big[\mu n_B(w)\bar{v}\big]j \\
&\quad + \big[\mu s(\bar{w})\overline{vw}\big] + \big[s(\bar{w})s(\overline{vw})1_B\big]j.
\end{aligned}$$

Summing up, we deduce

$$\begin{aligned}
(jv)(wj)^2 &= \big[-\mu n_B(w)s(v)1_B + \lambda\mu n_B(w)v + \mu s(\bar{w})\overline{vw}\big] \\
&\quad + \big[s(\bar{w})s(\overline{vw})1_B + \mu n_B(w)\bar{v}\big]j.
\end{aligned} \tag{12}$$

On the other hand, turning to the left-hand side of (10), we use (11), (3.2.4), (2.2.4), (2.2.6) to compute

$$\begin{aligned}\big((jv)(wj)\big)(wj) &= \mu\overline{vw}(wj) + s(\overline{vw})j(wj)\\ &= \big[-\mu s(\bar{w}\overline{vw})1_B + \mu s(\overline{vw})w + \mu s(\bar{w})\overline{vw} - \lambda\mu w\overline{vw}\big] + \big[\mu w\overline{vw}\big]j\\ &\quad + \big[\mu s(\overline{vw})\bar{w}\big] + \big[s(\bar{w})s(\overline{vw})1_B\big]j\\ &= \big[-\mu t_B(w)s(\overline{vw})1_B + \mu s(w\overline{vw})1_B + \mu s(\overline{vw})w + \mu s(\bar{w})\overline{vw}\\ &\quad - \lambda\mu w\overline{vw} + \mu s(\overline{vw})\bar{w}\big] + \big[\mu w\overline{vw} + s(\bar{w})s(\overline{vw})1_B\big]j\\ &= \big[-\mu t_B(w)s(\overline{vw})1_B + \mu t_B(w)s(\overline{vw})1_B + \mu n_B(w)s(\bar{v})1_B\\ &\quad + \mu s(\bar{w})\overline{vw} - \lambda\mu n_B(w)\bar{v}\big] + \big[\mu n_B(w)\bar{v} + s(\bar{w})s(\overline{vw})1_B\big]j\\ &= \big[\lambda\mu t_B(v)n_B(w)1_B - \mu n_B(w)s(v)1_B - \lambda\mu t_B(v)n_B(w)1_B\\ &\quad + \lambda\mu n_B(w)v + \mu s(\bar{w})\overline{vw}\big] + \big[s(\bar{w})s(\overline{vw})1_B + \mu n_B(w)\bar{v}\big]j\\ &= \big[-\mu n_B(w)s(v)1_B + \lambda\mu n_B(w)v + \mu s(\bar{w})\overline{vw}\big]\\ &\quad + \big[s(\bar{w})s(\overline{vw})1_B + \mu n_B(w)\bar{v}\big]j.\end{aligned}$$

Comparing with (12) gives (10).

6^0. We now proceed to verify (6) and first manipulate the left-hand side.

$$\begin{aligned}[u, vj, w] + [vj, u, w] &= \big(u(vj)\big)w - u\big((vj)w\big) + \big((vj)u\big)w - (vj)(uw)\\ &= \big(u\circ(vj)\big)w - u\big((vj)w\big) - (vj)(uw).\end{aligned}$$

Thus (6) is equivalent to

$$\big(u\circ(vj)\big)w - u\big((vj)w\big) - (vj)(uw) = \overline{H(u,\bar{v},w)}. \tag{13}$$

We begin by using (3.3.6), (3.2.4) to compute the first summand on the left-hand side of (13).

$$\begin{aligned}\big(u\circ(vj)\big)w &= \Big(\big[-s(\bar{v}u)1_B + s(\bar{v})u\big] + \big[t_B(u)v\big]j\Big)w\\ &= \big[-s(\bar{v}u)w + s(\bar{v})uw\big] + t_B(u)(vj)w\\ &= \big[-s(\bar{v}u)w + s(\bar{v})uw - t_B(u)s(w)v + \lambda t_B(u)vw\big] + \big[t_B(u)v\bar{w}\big]j\end{aligned}$$

Hence

$$\big(u\circ(vj)\big)w = \big[-t_B(u)s(w)v - s(\bar{v}u)w + s(\bar{v})uw + \lambda t_B(u)vw\big] + \big[t_B(u)v\bar{w}\big]j \tag{14}$$

Next we turn to the second summand on the left of (13) by using (3.2.4), (3.2.3) to compute

$$\begin{aligned}u\big((vj)w\big) &= u\Big(\big[-s(w)v + \lambda vw\big] + \big[v\bar{w}\big]j\Big) = \big[-s(w)uv + \lambda uvw\big] + u\big((v\bar{w})j\big)\\ &= \big[-s(w)uv + \lambda uvw - s(w\bar{v}u)1_B + s(u)v\bar{w} + s(w\bar{v})u - \lambda v\bar{w}u\big] + \big[v\bar{w}u\big]j,\end{aligned}$$

which amounts to

$$\begin{aligned}u\big((vj)w\big) &= \big[-s(w\bar{v}u)1_B + s(w\bar{v})u - s(w)uv + s(u)v\bar{w} + \lambda(uvw - v\bar{w}u)\big]\\ &\quad + \big[v\bar{w}u\big]j.\end{aligned} \tag{15}$$

Computing the third summand on the left of (13) is easy since (3.2.4) yields

$$(vj)(uw) = \big[-s(uw)v + \lambda vuw\big] + \big[v\bar{w}\bar{u}\big]j. \tag{16}$$

Putting things together in (14)−(16), we are able to compute the left-hand side of (13) as

$$\begin{aligned}\big(u\circ(vj\big)w - u\big((vj)w\big) - (vj)(uw) &= \big[-t_B(u)s(w)v - s(\bar{v}u)w + s(\bar{v})uw\\ &\quad + \lambda t_B(u)vw + s(w\bar{v}u)1_B - s(w\bar{v})u + s(w)uv - s(u)v\bar{w} - \lambda uvw\\ &\quad + \lambda v\bar{w}u + s(uw)v - \lambda vuw\big] + \big[t_B(u)v\bar{w} - v\bar{w}u - v\bar{w}\bar{u}\big]j.\end{aligned}$$

Here the Bj-component vanishes because of (2.2.6), so the entire expression belongs to B. Reordering we conclude

$$\begin{aligned}\big(u\circ(vj)\big)w-u\big((vj)w\big)-(vj)(uw)&=s(w\bar{v}u)1_B-s(w\bar{v})u-\big[t_B(u)s(w)-s(uw)\big]v\\&\quad-s(\bar{v}u)w+s(w)uv+s(\bar{v})uw+\lambda t_B(u)vw-s(u)v\bar{w}\\&\quad-\lambda(uvw-v\bar{w}u+vuw).\end{aligned}$$

Here $t_B(u)s(w)-s(uw)=s(\bar{u}w)$, while (2.2.4), (2.2.5) yield

$$\begin{aligned}uvw-v\bar{w}u+vuw&=uvw-t_B(w)vu+v(wu+uw)\\&=uvw-t_B(w)vu+t_B(u)vw+t_B(w)vu-n_B(u,w)v\\&=uvw+t_B(u)vw-t_B(\bar{u}w)v.\end{aligned}$$

since $t_B(\bar{u}w)-n_B(u,w)=t_B(\bar{u}w)-n_B(\bar{u},\bar{w})\in\mathrm{Ann}(B)$ by [16, Prop. 18.8 (a)]. Hence we obtain

$$\begin{aligned}\big(u\circ(vj)\big)w-u\big((vj)w\big)-(vj)(uw)&=s(w\bar{v}u)1_B-s(w\bar{v})u-s(\bar{u}w)v-s(\bar{v}u)w\\&\quad+s(w)uv+s(\bar{v})uw+\lambda t_B(u)vw-s(u)v\bar{w}\\&\quad-\lambda uvw-\lambda t_B(u)vw+\lambda t_B(\bar{u}w)v,\end{aligned}$$

where the term $\lambda t_B(u)vw$ cancels out and $-s(\bar{u}w)+\lambda t_B(\bar{u}w)=s(\overline{\bar{u}w})=s(\bar{w}u)$. Thus the left-hand side of (13) attains the final form

$$\begin{aligned}\big(u\circ(vj)\big)w-u\big((vj)w\big)-(vj)(uw)&=s(w\bar{v}u)1_B-s(w\bar{v})u+s(\bar{w}u)v-s(\bar{v}u)w\\&\quad+s(w)uv+s(\bar{v})uw-s(u)v\bar{w}-\lambda uvw\end{aligned}\tag{17}$$

Subtracting

$$\begin{aligned}\overline{H(u,\bar{v},w)}=-\overline{H(w,\bar{v},u)}&=s(w\bar{v}u)1_B-s(w\bar{v})u-s(\bar{v}u)w+s(w\bar{u})v\\&\quad-s(w)\bar{u}v+s(\bar{v})uw+s(u)vw-\lambda uvw,\end{aligned}$$

from the right-hand side of (17), we obtain

$$\begin{aligned}\big(s(\bar{w}u)-s(w\bar{u})\big)v+&s(w)(u+\bar{u})v-s(u)v(w+\bar{w})\\&=\big(t_B(w)s(u)-s(wu)-t_B(u)s(w)+s(wu)\\&\quad+t_B(u)s(w)-t_B(w)s(u)\big)v=0,\end{aligned}$$

which completes the proof of (13).

7^0. Finally, we will deal with (7), whose left-hand side may be written as

$$\begin{aligned}[wj,u,vj]+[wj,vj,u]&=\big((wj)u\big)(vj)-(wj)\big(u(vj)\big)+\big((wj)(vj)\big)u-(wj)\big((vj)u\big)\\&=\big((wj)u\big)(vj)+\big((wj)(vj)\big)u-(wj)\big(u\circ(vj)\big).\end{aligned}$$

Hence (7) is equivalent to

$$\big((wj)u\big)(vj)+\big((wj)(vj)\big)u-(wj)\big(u\circ(vj)\big)=-H(\bar{u},\bar{v},w)j.\tag{18}$$

In order to prove this, we begin with the last summand on the left, which by (3.3.6), (3.2.4) may be written as

$$\begin{aligned}(wj)\big(u\circ(vj)\big)&=(wj)\Big(\big[-s(\bar{v}u)1_B+s(\bar{v})u\big]+\big[t_B(u)v\big]j\Big)\\&=\big[-s(\bar{v}u)w\big]j+s(\bar{v})(wj)u+t_B(u)(wj)(vj)\\&=\big[-s(\bar{v}u)w\big]j+\big[-s(u)s(\bar{v})w+\lambda s(\bar{v})wu\big]+\big[s(\bar{v})w\bar{u}\big]j\\&\quad+t_B(u)(wj)(vj),\end{aligned}$$

which may be condensed to

$$\begin{aligned}(wj)\big(u\circ(vj)\big)&=\big[-s(u)s(\bar{v})w+\lambda s(\bar{v})wu\big]+\big[-s(\bar{v}u)w+s(\bar{v})w\bar{u}\big]j\\&\quad+t_B(u)(wj)(vj).\end{aligned}$$

Hence (18) may be written in the form

$$-H(\bar{u},\bar{v},w)j = \big[s(u)s(\bar{v})w - \lambda s(\bar{v})wu\big] + \big[s(\bar{v}u)w - s(\bar{v})w\bar{u}\big]j \\ + \big((wj)u\big)(vj) - \big((wj)(vj)\big)\bar{u}. \tag{19}$$

Our next aim will be to compute the final two terms on the right of (19). Using (3.2.4), (3.2.3), (3.2.5), we first obtain

$$\begin{aligned}\big((wj)u\big)(vj) &= \Big(\big[-s(u)w + \lambda wu\big] + \big[w\bar{u}\big]j\Big)(vj) \\ &= -s(u)w(vj) + \lambda(wu)(vj) + \big((w\bar{u})j\big)(vj) \\ &= \big[s(u)s(\bar{v}w)1_B - s(u)s(w)v - s(u)s(\bar{v})w + \lambda s(u)vw\big] + \big[-s(u)vw\big]j \\ &\quad + \big[-\lambda s(\bar{v}wu)1_B + \lambda s(wu)v + \lambda s(\bar{v})wu - \lambda^2 vwu\big] + \big[\lambda vwu\big]j \\ &\quad + \big[-\lambda s(\bar{v}w\bar{u})1_B + \lambda s(w\bar{u})v + \lambda s(\bar{v})w\bar{u} - \lambda^2 vw\bar{u} + \mu\bar{v}w\bar{u}\big] \\ &\quad + \big[s(\bar{v}w\bar{u})1_B - s(w\bar{u})v + \lambda vw\bar{u}\big]j \\ &= \big[\big(s(u)s(\bar{v}w) - \lambda t_B(u)s(\bar{v}w)\big)1_B - \big(s(u)s(w) - \lambda t_B(u)s(w)\big)v \\ &\quad - \big(s(u)s(\bar{v}) - \lambda t_B(u)s(\bar{v})\big)w + \lambda\big(s(u) - \lambda t_B(u)\big)vw + \mu\bar{v}w\bar{u}\big] \\ &\quad + \big[s(\bar{v}w\bar{u})1_B - s(w\bar{u})v - \big(s(u) - \lambda t_B(u)\big)vw\big]j\end{aligned}$$

Summing up, we therefore have

$$\big((wj)u\big)(vj) = \big[-s(\bar{u})s(\bar{v}w)1_B + s(\bar{u})s(w)v + s(\bar{u})s(\bar{v})w - \lambda s(\bar{u})vw + \mu\bar{v}w\bar{u}\big] \\ + \big[s(\bar{v}w\bar{u})1_B - s(w\bar{u})v + s(\bar{u})vw\big]j. \tag{20}$$

Next we tackle the very last term of (19) by using (3.2.5), (3.2.4) to obtain

$$\begin{aligned}\big((wj)(vj)\big)\bar{u} &= \Big(\big[-\lambda s(\bar{v}w)1_B + \lambda s(w)v + \lambda s(\bar{v})w - \lambda^2 vw + \mu\bar{v}w\big] \\ &\quad + \big[s(\bar{v}w)1_B - s(w)v + \lambda vw\big]j\Big)\bar{u} \\ &= \big[-\lambda s(\bar{v}w)\bar{u} + \lambda s(w)v\bar{u} + \lambda s(\bar{v})w\bar{u} - \lambda^2 vw\bar{u} + \mu\bar{v}w\bar{u}\big] \\ &\quad + \big(\big[s(\bar{v}w)1_B - s(w)v + \lambda vw\big]j\big)\bar{u} \\ &= \big[-\lambda s(\bar{v}w)\bar{u} + \lambda s(w)v\bar{u} + \lambda s(\bar{v})w\bar{u} - \lambda^2 vw\bar{u} + \mu\bar{v}w\bar{u}\big] \\ &\quad + \big[-s(\bar{u})s(\bar{v}w)1_B + s(\bar{u})s(w)v - \lambda s(\bar{u})vw + \lambda s(\bar{v}w)\bar{u} \\ &\quad - \lambda s(w)v\bar{u} + \lambda^2 vw\bar{u}\big] + \big[s(\bar{v}w)u - s(w)vu + \lambda vwu\big]j.\end{aligned}$$

In the B-component of this expression, the terms $\lambda s(\bar{v}w)\bar{u}$, $\lambda s(w)v\bar{u}$, $\lambda^2 vw\bar{u}$ cancel out, and what remains is

$$\big((wj)(vj)\big)\bar{u} = \big[-s(\bar{u})s(\bar{v}w)1_B + s(\bar{u})s(w)v + \lambda s(\bar{v})w\bar{u} - \lambda s(\bar{u})vw + \mu\bar{v}w\bar{u}\big] \\ + \big[s(\bar{v}w)u - s(w)vu + \lambda vwu\big]j. \tag{21}$$

With the aid of (20), (21), the right-hand side of (19) now attains the value

$$\begin{aligned}x := \big[&s(u)s(\bar{v})w - \lambda s(\bar{v})wu - s(\bar{u})s(\bar{v}w)1_B + s(\bar{u})s(w)v + s(\bar{u})s(\bar{v})w \\ &- \lambda s(\bar{u})vw + \mu\bar{v}w\bar{u} + s(\bar{u})s(\bar{v}w)1_B - s(\bar{u})s(w)v - \lambda s(\bar{v})w\bar{u} \\ &+ \lambda s(\bar{u})vw - \mu\bar{v}w\bar{u}\big] + \big[s(\bar{v}u)w - s(\bar{v})w\bar{u} + s(\bar{v}w\bar{u})1_B \\ &- s(w\bar{u})v + s(\bar{u})vw - s(\bar{v}w)u + s(w)vu - \lambda vwu\big]j.\end{aligned} \tag{22}$$

In the B-component of x, the terms $s(\bar{u})s(\bar{v}w)1_B$, $s(\bar{u})s(w)v$, $\lambda s(\bar{u})vw$ and $\mu\bar{v}w\bar{u}$ cancel out; hence it reduces to

$$s(u)s(\bar{v})w + s(\bar{u})s(\bar{v})w - \lambda s(\bar{v})w(u + \bar{u}) = \lambda t_B(u)s(\bar{v})w - \lambda t_B(u)s(\bar{v})w = 0,$$

and we conclude that the B-component of x is zero. On the other hand, using (7.1.1) and (22) to compare the Bj-component of x with

$$\begin{aligned}-H(\bar{v}, w, \bar{u}) &= s(\bar{v}w\bar{u})1_B - s(\bar{v}w)u - s(w\bar{u})v + s(\bar{v}u)w \\ &\quad - s(\bar{v})w\bar{u} + s(w)vu + s(\bar{u})vw - \lambda vwu,\end{aligned}$$

we see that they are the same. Moreover, since H is alternating, $H(\bar{v}, w, \bar{u}) = H(\bar{u}, \bar{v}, w)$, which completes the proof of (19), hence of (7) as well. This also completes the proof of the theorem. ☐

7.5. **Corollary.** *C is alternative if and only if B is associative, the conjugation of B is an involution, and $H_{B,s} = 0$.*

Proof. If B is associative, its conjugation is an involution and $H_{B,s} = 0$, then C is left alternative by Thm. 7.4, hence also right alternative since its conjugation by Cor. 4.5 is an involution. Thus C is alternative. Conversely, let this be so. Again by Thm. 7.4, it suffices to show that B is associative and its conjugation is an involution. Since C is flexible, the latter follows from Thm. 4.12. To prove the former, consider the equation

$$\big(u \circ (vj)\big)w = u\big((vj)w\big) + (vj)(uw), \tag{1}$$

valid for all $u, v, w \in B$ by alternativity of C. More precisely, we compare the Bj-components of both sides of (1) by computing $\bmod B$. Indeed, applying (3.3.6), (3.2.4), (3.2.3), we obtain

$$\begin{aligned}\big(u \circ (vj)\big)w &= \big(\big[-s(\bar{v}u)1_B + s(\bar{v})u\big] + \big[t_B(u)v\big]j\big)w \\ &\equiv t_B(u)(vj)w \equiv (t_B(u)v\bar{w})j \bmod B, \\ u\big((vj)w\big) &= u\big(\big[-s(w)v + \lambda vw\big] + \big[v\bar{w}\big]j\big) \\ &\equiv u\big((v\bar{w})j\big) \equiv \big[(v\bar{w})u\big]j \bmod B, \\ (vj)(uw) &\equiv \big[v\overline{uw}\big]j \equiv \big[v(\bar{w}\bar{u})\big]j \bmod B.\end{aligned}$$

Comparing and applying (1), we conclude

$$t_B(u)v\bar{w} = (v\bar{w})u + v(\bar{w}\bar{u}) = (v\bar{w})u + t_B(u)v\bar{w} - v(\bar{w}u),$$

which amounts to $[v, \bar{w}, u] = 0$ and hence shows that B is associative. ☐

7.6. **Corollary.** *If B is associative, its conjugation is an involution and B is locally generated by two elements (e.g., a quaternion algebra over k), then C is alternative.*

Proof. This follows by simply combining Cor. 7.5 with Cor. 7.3. ☐

7.7. **Concluding remarks.** It would be nice to have a version of Thm. 7.4 and its corollaries also in the commutative (resp. the associative) case. In other words, one would like to have

(i) if B is commutative, a formula for the commutator $[x, y]$, $x, y \in C$, in terms of an appropriate alternating bilinear map, to be bult up along the lines of the condition stated in Prop. 5.5 (ii),
(ii) if B is commutative associative, a formula for the associator $[x, y, z]$, $x, y, z \in B$ in terms of an alternating trilinear map, to be built up along the lines of the condition described in (6.6.1).

8. Non-singularity

In this section, we will be concerned with conditions under which the property of a conic algebra to be (weakly) non-singular is preserved by the non-orthogonal Cayley-Dickson construction. We also present a criterion that guarantees the output of a non-orthogonal Cayley-Dickson construction to be non-singular even though the conic algebra entering into the construction was only weakly non-singular to begin with. Using this criterion, we recover the octonionic structure exhibited by Coxeter [2] on the E_8-lattice.

Throughout we fix an arbitrary commutative ring k and a conic algebra B over k.

8.1. **Non-singular and weakly non-singular conic algebras.** (a) Our conic k-algebra B is said to be *non-singular* if its norm, n_B, is a non-singular quadratic form in the sense of [16, 12.11]. This means that B is finitely generated projective as a k-module and the bilinearization of the norm, more specifically written as

$$Dn_B\colon\ B \times B \longrightarrow k, \quad (u,v) \longmapsto (Dn_B)(u,v) := n_B(u+v) - n_B(u) - n_B(v),$$

induces a linear isomorphism from the k-module B onto its dual in the usual way. The importance of this concept derives from the fact that it is invariant under base change. By [16, Thm. 21.8], non-singular alternative conic algebras are the same as non-singular composition algebras.

(b) In a more general vein, B is said to be *weakly non-singular* if it is finitely generated projective as a k-module and n_B is a *weakly non-singular* quadratic form in the sense that the natural linear map from the k-module B to its dual induced by Dn_B is injective. Following [16, 12.3] to define the *radical* of Dn_B by

$$\mathrm{Rad}(Dn_B) := \big\{u \in B \mid n_B(u,B) = \{0\}\big\} \subseteq B, \tag{1}$$

we see that n_B is weakly non-singular if and only if $\mathrm{Rad}(Dn_B) = \{0\}$. Though no longer invariant under base change, the notion of weak non-singularity for conic algebras turns out to be quite useful in the present context.

8.2. **Lemma.** *Let M, M' be finitely generated projective k-modules and suppose we are given quadratic forms $q\colon M \to k$, $q'\colon M' \to k$ over k. With $\alpha \in k$, the following conditions are equivalent.*

(i) *The quadratic form $q \oplus (\alpha q')\colon M \oplus M' \to k$ is non-singular (resp. weakly non-singular).*
(ii) *q and q' are non-singular (resp. weakly non-singular) and α is invertible (resp. not a zero divisor) in k.*

Proof. Following [16, 10.4], we denote by $(x^*, x) \mapsto \langle x^*, x\rangle$ the canonical pairing $M^* \times M \to k$ (ditto for M') and write $\varphi\colon M \to M^*$ (resp. $\varphi'\colon M' \to M'^*$) for the natural map induced by Dq (resp. Dq'). Identifying $(M \oplus M')^* = M^* \oplus M'^*$ canonically via

$$\langle x^* \oplus x'^*, x \oplus x'\rangle = \langle x^*, x\rangle + \langle x'^*, x'\rangle \qquad (x \in M,\ x' \in M',\ x^* \in M^*,\ x'^* \in M'^*),$$

it is then straightforward to check that the linear map $M \oplus M' \to (M \oplus M')^* = M^* \oplus M'^*$ corresponding to $q \oplus (\alpha q')$ agrees with $\varphi \oplus \alpha\varphi'$. In view of [16, 20.7], the equivalence of (i), (ii) follows from this at once. $\square$

8.3. **Proposition.** *Assume that B is weakly non-singular. Let $\mu \in k$, $a \in B$,*

$$s := n_B(a, -)$$

as a linear form on B and put $C := \mathrm{Cay}(B; \mu, s)$. Then the following statements hold.

(a) *If B is flexible, then*

$$B^\perp = \{-va + vj \mid v \in B\}. \tag{1}$$

is its orthogonal complement in C relative to Dn_C. In particular, $j_0 := -a + j \in B^\perp$ and

$$\mu' := -n_C(j_0) = n_B(a) + \mu. \tag{2}$$

(b) *If B is alternative then*

$$\mathrm{Rad}(Dn_C) = \big\{-va + vj \mid \big(n_B(a) + \mu\big)v = 0\big\}. \tag{3}$$

Proof. By [16, Prop. 18.12], B is norm-associative and its conjugation is an involution.
(a) For $u, u', v \in B$, we use (3.3.3), (2.3.4) to compute

$$\begin{aligned} n_C(u + vj, u') &= n_B(u,u') + s(\bar{v}u') = n_B(u,u') + n_B(a, \bar{v}u') \\ &= n_B(u,u') + n_B(va, u') = n_B(u + va, u'). \end{aligned}$$

Hence, as B is weakly non-singular, $u + vj$ belongs to $B^\perp$ if and only if $u + va = 0$, equivalently, $u = -va$. This proves (1) and then immediately implies $j_0 \in B^\perp$. Moreover, by (3.3.2),

$$\begin{aligned} n_C(j_0) &= n_C(-a + j) = n_B(a) + s(-a) - \mu = n_B(a) - n_B(a, a) - \mu \\ &= n_B(a) - 2n_B(a) - \mu = -\big(n_B(a) + \mu\big), \end{aligned}$$

giving (2).

(b) Let us now assume that B is alternative. An element $x = u + vj \in C$, $u, v \in B$, belongs to $\mathrm{Rad}(Dn_C)$ if and only if $x \in B^\perp$ and $n_C(x, wj) = 0$ for all $w \in B$, which by (1) and (3.3.3) is equivalent to $x = -va + vj$ for some $v \in B$ and

$$\begin{aligned} 0 = n_C(-va + vj, wj) =&\ - s\big((\bar{w}(va)\big) - \mu n_B(v, w) = -n_B(a, \bar{w}(va)) - \mu n_B(v, w) \\ =&\ - n_B(a\overline{va}, \bar{w}) - \mu n_B(v, w) = -n_B\big(a(\bar{a}\bar{v}), \bar{w}\big) - \mu n_B(v, w) \\ =&\ - n_B(a)n_B(\bar{v}, \bar{w}) - \mu n_B(v, w) = -n_B\Big(\big(n_B(a) + \mu\big)v, w\Big) \end{aligned}$$

for all $w \in B$. Hence, again by weak non-singularity of B, equation (3) follows. $\square$

8.4. **Corollary.** *With the notation of Prop.* 8.3, *let us assume that B is weakly non-singular and alternative.*

(a) *C is weakly non-singular if and only if $\mu' = n_B(a) + \mu$ is not a zero divisor in k. In this case, there is a unique homomorphism*

$$\varphi\colon\ C' := \mathrm{Cay}(B, \mu') = B + Bj' \longrightarrow C$$

of conic algebras extending the identity of B and sending j' to j_0. Moreover, φ is an isomorphism.

(b) *C is non-singular if and only if B is non-singular and $\mu' = n_B(a) + \mu$ is invertible in k.*

Proof. (a) By [16, 20.7], μ_0 is not a zero divisor in k if and only if it isn't one in B. Hence the first assertion follows immediately from (8.3.3). Existence and uniqueness of the homomorphism φ may be read off directly from [16, Prop. 20.6], while [16, Cor. 20.8] shows that φ is an isomorphism from C' onto the subalgebra of C generated by B and j_0. But this subalgebra contains $j = a + j_0$ and hence agrees with all of C, so φ must be an isomorphism.

(b) If C is non-singular, then so is C' by (a), forcing B to be non-singular and μ' to be invertible in k since Lemma 8.2 applies to $n_{C'} \cong n_B \oplus (-\mu')n_B$. Conversely, suppose B is non-singular and μ' is invertible in k. Since, for the same reason as before, C' is non-singular, so is C by (a), and (b) follows. $\square$

8.5. **Remark.** The preceding results show that, under very peculiar circumstances, the property of a conic algebra to be (weakly) non-singular is preserved by the non-orthogonal Cayley-Dickson construction. But with an eye on Cor. 8.4 (a), one is tempted to ask: what's the point of the non-orthogonal Cayley-Dickson construction when restricting oneself to situations where everything can be blamed on the orthogonal one? It is therefore important to exhibit instances of the non-orthogonal Cayley-Dickson construction where the reduction to the orthogonal case is no longer possible. Such instances may already be found in [5, Prop. 4.4], where the non-orthogonal Cayley-Dickson construction leads from a purely inseparable field extension of characteristic two and exponent one to a weakly non-singular conic algebra. We will now proceed to exhibit instances of this kind that are no longer restricted to base fields of characteristic two but, in fact, work over arbitrary integral domains.

8.6. **Lemma.** *Let k be an integral domain with quotient field $K := \mathrm{Quot}(k)$ and suppose B is weakly non-singular. Then the following statements hold.*

(a) *B_K is a non-singular conic algebra over K.*

(b) *The natural map $B \to B_K$ is injective.*

(c) *Identifying $B \subseteq B_K$ by means of* (b),

$$B^\sharp := \{w \in B_K \mid n_{B_K}(w, B) \subseteq k\}$$

is a finitely generated projective k-submodule of B_K satisfying $B \subseteq B^\sharp$.

(d) *There exists a non-zero element $\delta \in k$ such that $B^\sharp \subseteq \delta^{-1}B$.*

(e) *If B is flexible, then $BB^\sharp + B^\sharp B = B^\sharp$.*

Proof. (a) The natural k-linear map $\varphi\colon B \to B^*$ determined by Dn_B is injective by hypothesis. Hence so is its K-linear extension $\varphi_K\colon B_K \to (B^*)_K = (B_K)^*$ [16, 10.6] since K is a flat k-algebra [1, II, §2, Thm. 1]. Thus B_K is weakly non-singular. But K is a field and B_K is a finite-dimensional over K, forcing it to be, in fact, a non-singular conic K-algebra.

(b) Given $u \in B$ such that $u/1 \in B_K$ is zero, then some non-zero $\delta \in k$ has $\delta u = 0$. But δ, not being a zero divisor in k, neither is one in B [16, 20.7]. Hence $u = 0$.

(c) $B^\sharp \subseteq B_K$ is clearly a k-submodule containing B, so it suffices to show that it is finitely generated projective. Localizing if necessary, we may assume that B is free as a k-module, of rank n, say. Accordingly, let $(e_i)_{1\leq i\leq n}$ be a k-basis of B, which is automatically a K-basis of B_K. Since B_K is non-singular over K by (a), we may consider the corresponding dual basis $(e_i^*)_{1\leq i\leq n}$ of B_K over K relative to Dn_{B_K}, and it is straightforward to check that $B^\sharp$ is the free k-module generated by the e_i^*, $1 \leq i \leq n$. This completes the proof of (c).

(d) This follows immediately from the fact that, by (c), $B^\sharp$ is finitely generated as a k-module.

(e) Let $u, v \in B$ and $w \in B^\sharp$. By (2.3.4), (2.3.3) we have $n_{B_K}(vw, u) = n_{B_K}(w, \bar{v}u) \in k$, forcing $vw \in B^\sharp$, and $n_{B_K}(wv, u) = n_{B_K}(w, u\bar{v}) \in k$, forcing $wv \in B^\sharp$. □.

8.7. **Proposition.** *With the notation and assumptions of Lemma* 8.6, *B is non-singular if and only if $B = B^\sharp$.*

Proof. Suppose first that B is non-singular. By Lemma 8.6 (c), we must show $B^\sharp \subseteq B$, so let $w \in B^\sharp$. Then the map $B \to k$, $u \mapsto n_{B_K}(w, u)$ is a linear form, which by non-singularity of B can be written as $n_B(a, -)$ for some $a \in B$. Since $KB = B_K$, we conclude $w - a \in \mathrm{Rad}(Dn_{B_K}) = \{0\}$, hence $w = a \in B$. Conversely, suppose $B = B^\sharp$ and let $s\colon B \to k$ be a linear form. Since B_K is non-singular by Lemma 8.6 (a), the K-linear extension of s has the form $s_K = n_{B_K}(w, -)$ for some $w \in B_K$. In particular, $n_{B_K}(w, u) = s_K(u) = s(u) \in k$ for all $u \in B$, which implies $w \in B^\sharp = B$, hence $s = n_B(w, -)$, and by weak non-singularity of B we are done. □

8.8. **Lemma.** *With the notation and assumptions of Lemma* 8.6, *let $\mu \in k \setminus \{0\}$ and $C := \mathrm{Cay}(B, \mu) = B \oplus Bj$. Then*

$$C^\sharp = B^\sharp \oplus (\mu^{-1}B^\sharp)j.$$

Proof. Write $x \in C_K$ uniquely in the form $x = a + bj$ for some $a, b \in B_K$. In view of Lemma 8.6 (c), we then obtain the following chain of equivalent conditions.

$$\begin{aligned} x \in C^\sharp &\Longleftrightarrow \forall u, v \in B: \ n_{C_K}(a + bj, u) \in k, \ n_{C_K}(a + bj, vj) \in k \\ &\Longleftrightarrow \forall u, v \in B: \ n_{B_K}(a, u) \in k, \ n_{B_K}(\mu b, v) = \mu n_{B_K}(b, v) \in k \\ &\Longleftrightarrow a \in B^\sharp, \ \mu b \in B^\sharp \\ &\Longleftrightarrow a \in B^\sharp, \ b \in \mu^{-1}B^\sharp \\ &\Longleftrightarrow x = a + bj \in B^\sharp \oplus (\mu^{-1}B^\sharp)j. \end{aligned}$$

The assertion follows. □

Trying to find an analogous description in the setting of the non-orthogonal Cayley-Dickson construction, turns out to be a bit more troublesome.

8.9. **Proposition.** *With the notation and assumptions of Lemma* 8.6, *let* $\mu \in k$ *and* $s\colon B \to k$ *a linear form. Then there exists a unique element* $a \in B_K$ *such that* $s_K = n_{B_K}(a, -)$. *Moroever, if* B *is flexible, we put* $C := \mathrm{Cay}(B; \mu, s) = B \oplus Bj$ *and have*

$$C^\sharp = \{u + vj \mid u, v \in B_K, \ u + va, u\bar{a} - \mu v \in B^\sharp\}.$$

Proof. Existence and uniqueness of a follows from the fact that B_K is non-singular over K (Lemma 8.6 (a)). Now suppose B is flexible, hence norm-associative [16, Prop. 18.12]. For $u, v, w \in B_K$, it suffices to show

$$(1) \qquad n_{C_K}(u + vj, w) = n_{B_K}(u + va, w), \quad n_{C_K}(u + vj, wj) = n_{B_K}(u\bar{a} - \mu v, w).$$

Indeed, by (3.3.3), (2.3.4) and the definition of a, we have

$$\begin{aligned} n_{C_K}(u + vj, w) &= n_{B_K}(u, w) + s_K(\bar{v}w) = n_{B_K}(u, w) + n_{B_K}(a, \bar{v}w) \\ &= n_{B_K}(u, w) + n_{B_K}(va, w) = n_{B_K}(u + va, w), \end{aligned}$$

giving the first equation of (1), and

$$\begin{aligned} n_{C_K}(u + vj, wj) &= s_K(\bar{w}u) - \mu n_{B_K}(v, w) = n_{B_K}(a, \bar{w}u) - \mu n_{B_K}(v, w) \\ &= n_{B_K}(a\bar{u}, \bar{w}) - \mu n_{B_K}(v, w) = n_{B_K}(u\bar{a}, w) - \mu n_{B_K}(v, w) \\ &= n_{B_K}(u\bar{a} - \mu v, w), \end{aligned}$$

giving the second equation of (1) as well. □

Remark. If C as in Prop. 8.9 is an orthogonal Cayley-Dickson construction, then $a = 0$, and we recover Lemma 8.8 in the flexible case.

8.10. **Theorem.** *Let* k *be an intergral domain with quotient field* $K := \mathrm{Quot}(k)$, B *a weakly non-singular conic alternative* k*-algebra,* $\mu \in k$ *an arbitrary scalar,* $s\colon B \to k$ *an arbitrary linear form and* $C := \mathrm{Cay}(B; \mu, s) = B \oplus Bj$ *as in* 3.2. *Choose an element* $\delta \in k \setminus \{0\}$ *satisfying* $B^\sharp \subseteq \delta^{-1}B$ (*cf. Lemma* 8.6 (d)) *and write* a *for the unique element in* B_K *satisfying* $s_K = n_{B_K}(a, -)$ (*cf. Prop.* 8.9). *Then the following statements hold.*

(a) $a \in B^\sharp$.

(b) *There exists a unique element* $a_0 \in B$ *such that* $a = \delta^{-1}a_0$.

(c) $\varepsilon := n_B(a_0) \in k$.

(d) *If* $\varepsilon + \delta^2\mu$ *is invertible in* k, *then* C *is non-singular.*

Proof. (a) We have $n_{B_K}(a, u) = s_K(u) = s(u) \in k$ for all $u \in B$, hence $a \in B^\sharp$

(b) This follows from the choice of δ.

(c) This is clear.

(d) By Prop. 8.7, we must show $C = C^\sharp$, so let $u, v \in B_K$ satisfy $u + vj \in C^\sharp$. By Prop. 8.9 and the choice of δ, this implies

$$(1) \qquad u + va, \ u\bar{a} - \mu v \in \delta^{-1}B.$$

Now observe $t_{B_K}(a) = n_{B_K}(a, 1_B) \in k$ by (a), and Lemma 8.6 (c) yields $\bar{a} = t_{B_K}(a)1_B - a \in B + B^\sharp = B^\sharp \subseteq \delta^{-1}B$ by (1). Thus (1) again combines with alternativity of B to imply

$$u\bar{a} + n_{B_K}(a)v = (u + va)\bar{a} \in \delta^{-2}B.$$

Since $n_{B_K}(a) = \delta^{-2}\varepsilon$ by (b), (c), we therefore conclude from (1) that

$$\delta^{-2}(\varepsilon + \delta^2\mu)v = (\delta^{-2}\varepsilon + \mu)v = \big(n_{B_K}(a) + \mu\big)v = \big(u\bar{a} + n_{B_K}(a)v\big) - (u\bar{a} - \mu v)$$

belongs to $\delta^{-2}B$ as well. But this forces $(\varepsilon+\delta^2\mu)v \in B$, hence $v \in B$ by hypothesis. Now the second equation of (1) implies $u\bar{a} \in \delta^{-1}B$, forcing $\delta^{-2}\varepsilon u = n_{B_K}(a)u = (u\bar{a})a \in \delta^{-2}B$, hence $\varepsilon u \in B$. On the other hand, since va belongs to $\delta^{-1}B$ by Lemma 8.6 (e), so does u by the first equation of (1). Thus $\delta^2\mu u \in \delta B \subseteq B$, and we end up with $(\varepsilon + \delta^2\mu)u \in B$. This gives $u \in B$ by hypothesis, and summing up we have proved $u + vj \in B + Bj = C$, as desired. $\square$

9. Examples: Hurwitz, Dickson and Coxeter

In this section we work over the ring $\mathbb{Z}$ of rational integers. Our aim is to describe the Hurwitz quaternions as well as the Dickson and Coxeter octonions as explicitly as possible by means of the non-orthogonal Cayley-Dickson construction. We will also show that certain unimodular positive definite integral quadratic lattices of rank 16 carry the structure of a sedenion algebra over the integers.

9.1. **Gaussian integers.** Let $\mathbb{D}$ be one of the four composition division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ over the reals. It is well known that $\mathbb{D}$ admits an orthonormal basis $E = (u_i)_{0\le i<n}$, $n = \dim_{\mathbb{R}}(D)$, relative to the scalar product $\langle x, y\rangle := \frac{1}{2}n_{\mathbb{D}}(x, y)$ that up to a sign is closed under multiplication. Then

$$\mathrm{Ga}_E(\mathbb{D}) := \bigoplus_{0\le i<n} \mathbb{Z}u_i$$

is a $\mathbb{Z}$-*structure* of $\mathbb{D}$, i.e., it is a subalgebra over $\mathbb{Z}$ such that the embedding $\mathrm{Ga}_E(\mathbb{D}) \hookrightarrow \mathbb{D}$ induces an isomorphism from $\mathrm{Ga}_E(\mathbb{D}) \otimes_{\mathbb{Z}} \mathbb{R}$ onto $\mathbb{D}$. We call $\mathrm{Ga}_E(\mathbb{D})$ the ring of *Gaussian integers* of $\mathbb{D}$ with respect to E. Note that $\mathrm{Ga}_E(\mathbb{D})$ is a weakly non-singular conic alternative algebra over $\mathbb{Z}$ satisfying

$$\mathrm{Ga}_E(\mathbb{D})^\sharp = \frac{1}{2}\mathrm{Ga}_E(\mathbb{D}) = \frac{1}{2}\bigoplus_{0\le i<n} \mathbb{Z}u_i \tag{1}$$

in the sense of Lemma 8.6 (c).

9.2. **Example: the classical Gaussian integers.** The preceding considerations apply to $D := \mathbb{C}$ and its real basis $E := (1, i)$. Then $\mathrm{Ga}(\mathbb{C}) := \mathrm{Ga}_E(\mathbb{C}) = \mathbb{Z}[i] = \mathrm{Cay}(\mathbb{Z}, -1)$ are the classical Gaussian integers, i.e., the ring of integers in the Gaussian number field $\mathbb{Q}(i)/\mathbb{Q}$.

9.3. **Example: the Hurwitz quaternions.** Let $\mathbb{H}$ be the algebra of Hamiltonian quaternions with its canonical basis $E := (1, \mathbf{i}, \mathbf{j}, \mathbf{k})$. The Hurwitz quaternions are traditionally written as

$$\mathrm{Hur}(\mathbb{H}) = \mathbb{Z}1 \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{h}, \quad \mathbf{h} := \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \tag{1}$$

as an algebra over $\mathbb{Z}$ [6, p. 319]. According to [16, Exc. 117], they cannot be realized by the orthogonal Cayley-Dickson construction. In order to realize them by the non-orthogonal one, we identify $B := \mathrm{Ga}(\mathbb{C}) = \mathbb{Z}[\mathbf{i}] \subseteq \mathbb{C} = \mathbb{R}[\mathbf{i}] \subseteq \mathbb{H}$ and apply [16, Thm. 4.2] to conclude

$$\mathrm{Hur}(\mathbb{H}) = \mathrm{Ga}(\mathbb{C}) \oplus \mathrm{Ga}(\mathbb{C})\mathbf{h} = \mathbb{Z}[\mathbf{i}] \oplus \mathbb{Z}[\mathbf{i}]\mathbf{h}. \tag{2}$$

Consulting Prop. 3.5 and observing $n_{\mathbb{H}}(\mathbf{h}) = 1$, we therefore obtain an identification

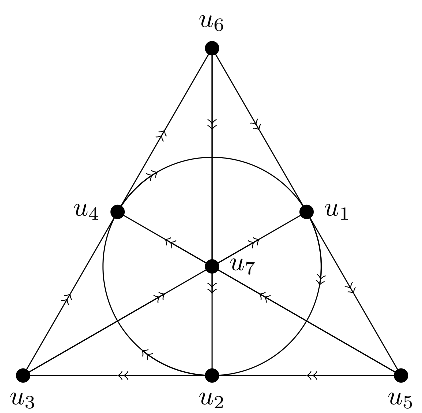
$$\mathrm{Hur}(\mathbb{H}) = \mathrm{Cay}\big(\mathbb{Z}[\mathbf{i}]; s, -1\big), \tag{3}$$

where $s\colon \mathbb{Z}[\mathbf{i}] \to \mathbb{Z}$ is the linear form that in view of (1) may be written as

$$s(v) = n_{\mathbb{H}}(\mathbf{h}, v) = n_{B_{\mathbb{Q}}}(a, v), \quad a := \frac{1}{2}a_0 \in B^\sharp, \quad a_0 := 1 + \mathbf{i} \in B \qquad (v \in B) \tag{4}$$

since $\mathbf{j}, \mathbf{k} \in B^\perp \subseteq \mathrm{Hur}(\mathbb{H})$; thus $s(1) = s(\mathbf{i}) = 1$. Adopting the notation of Thm. 8.10 we have $\mu = -1$, $\delta = \varepsilon = 2$, hence $\varepsilon + \delta^2\mu = -2 \notin \mathbb{Z}^\times$, in agreement with Thm. 8.10 (d) since the Hurwitz quaternions are known to be singular, with discriminant 4 [16, Thm. 4.2].

9.4. **Cartan-Shouten bases.** As usual, we denote by $\mathbb{O}$ the real algebra of Graves-Cayley octonions [16, 1.4] and, following [16, 2.1], define a *Cartan-Shouten basis* of $\mathbb{O}$ as a family $(u_i)_{0 \le i \le 7}$ of elements in $\mathbb{O}$ such that $u_0 = 1_{\mathbb{O}}$, $u_i^2 = -u_0$ and $u_{i+r}u_{i+3r} = u_i = -u_{i+3r}u_{i+r}$ for $1 \le i \le 7$ and $r = 1, 2, 4$, where indices are to be reduced $\bmod 7$ whenever necessary. Cartan-Shouten bases exist [16, Prop. 2.2] and are orthonormal bases of the underlying vector space [16, Exc. 6] which fit into the pattern of 9.1. More precisely, the multiplicative properties of Cartan-Shouten bases may be depicted conveniently in the



usual way by the visualization of the Fano plane, see [16, 2.5] for details.

9.5. **Example: the Coxeter octonions.** Let $E = (u_i)_{1 \le i \le 7}$ be a Cartan-Shouten basis of $\mathbb{O}$. A matching copy of $\mathbb{H}$ in $\mathbb{O}$ is obtained by identifying the canonical basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ of $\mathbb{H}$ with $E' := (u_0, u_1, u_2, u_4)$, so we have

$$\mathbb{H} = \mathbb{R}u_0 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}u_4, \tag{1}$$

and conclude from 9.1 that

$$B := \mathrm{Ga}_{E'}(\mathbb{H}) = \mathbb{Z}u_0 \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2 \oplus \mathbb{Z}u_4 \tag{2}$$

is a weakly non-songular associative conic algebra over the integers satisfying $B^\sharp = \frac{1}{2}B$, hence $\delta = 2$ in the sense of Thm. 8.10. Putting

$$\mathbf{p} := \frac{1}{2}(u_0 + u_1 + u_2 + u_3), \tag{3}$$

we now deduce from [16, Thm. 4.5] that

$$\mathrm{Cox}(\mathbb{O}) := B \oplus B\mathbf{p} \tag{4}$$

is a $\mathbb{Z}$-structure of $\mathbb{O}$, called the *Coxter octonions*, whose generic fibre is the unique octonion division algebra over the rationals. From [16, Exc. 16] we deduce that $(\varepsilon_i)_{1 \le i \le 8}$, with

$$\varepsilon_1 := \frac{1}{2}(-u_0 + u_2), \quad \varepsilon_2 := \frac{1}{2}(u_0 + u_2), \quad \varepsilon_3 := -\frac{1}{2}(u_1 + u_3), \quad \varepsilon_4 := \frac{1}{2}(u_1 - u_3),$$
$$\varepsilon_5 := \frac{1}{2}(-u_4 + u_5), \quad \varepsilon_6 := \frac{1}{2}(u_4 + u_5), \quad \varepsilon_7 := \frac{1}{2}(u_6 - u_7), \quad \varepsilon_8 := \frac{1}{2}(u_6 + u_7)$$

is an orthonormal basis of $\mathbb{O}$ and

$$\text{(5)} \qquad \mathrm{Cox}(\mathbb{O}) = \big\{ \sum_{i=1}^{8} \xi_i \varepsilon_i \mid \xi_i \in \mathbb{R},\ 2\xi_i, \xi_i - \xi_j \in \mathbb{Z}\ (1 \le i, j \le 8), \quad \sum_{i=1}^{8} \xi_i \in 2\mathbb{Z} \big\}.$$

Since $n_{\mathbb{O}}(\mathbf{p}) = 1$ by (3), we deduce from Prop. 3.5, (4) and (3) again that

$$\text{(6)} \qquad \mathrm{Cox}(\mathbb{O}) = \mathrm{Cay}(B; s, -1)$$

where $s\colon B \to \mathbb{Z}$ is the linear form given by

$$\text{(7)} \qquad s(v) = n_{\mathbb{O}}(\mathbf{p}, v) = n_{B_{\mathbb{Q}}}(a, v), \quad a := \frac{1}{2} a_0 \in B^\sharp, \quad a_0 := u_0 + u_1 + u_2 \qquad (v \in B)$$

since $u_3 \in B^\perp \subseteq \mathrm{Cox}(\mathbb{O})$; thus s is characterized by $s(u_i) = 1$ for $i = 0, 1, 2$ and $s(u_4) = 0$. In the notation of Thm. 8.10, we have $\mu = -1$, $\delta = 2$ and $\varepsilon = 3$, hence $\varepsilon + \delta^2\mu = -1 \in \mathbb{Z}^\times$, and we conclude from Thm. 8.10 (d) that $\mathrm{Cox}(\mathbb{O})$ is an octonion algebra over the integers. Note that we have arrived at this conclusion without computing any determinants; simply invoking the aforementioned general theorem was enough.

Alternatively, starting from B, the Gaussian integers of $\mathbb{H}$ relative to the canonical basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ of 9.3, and the linear form $s := n_{\mathbb{H}}(a, -)\colon B \to \mathbb{Z}$ with $a := \frac{1}{2}(1 + \mathbf{i} + \mathbf{j}) \in B^\sharp$ as in (7), we could have defined the Coxeter octonions by the right-hand side of (6), which the arguments of the preceding § establish as an octonion algebra over the integers whose generic fiber, by Cor. 8.4, is isomorphic to $\mathrm{Cay}(B_{\mathbb{Q}}, -\frac{1}{4})$, hence to the unique octonion division algebra over the rationals.

9.6. **Towards the Dickson octonions.** The Dickson octonions as defined in [3] (see also Mahler [11]), which form an octonion algebra without zero divisors over the integers and hence, by a result of Van der Blij-Springer [18], are isomorphic to the Coxeter octonions, will be shown in this section to arise from the Hurwitz quaternions by means of the non-orthogonal Cayley-Dickson construction.

As a first step, the same objective will be achieved for the Coxeter octonions. In order to do so, we pick a Cartan-Shouten basis $E = (u_i)_{0 \le i \le 7}$ of $\mathbb{O}$ as in 9.5. Identifying $\mathbb{H}$ in $\mathbb{O}$ via (9.5.1) won't do since (9.5.5) is easily seen to imply that $\mathbf{h} = \frac{1}{2}(u_0 + u_1 + u_2 + u_4)$ does *not* belong to $\mathrm{Cox}(\mathbb{O})$; hence neither does $\mathrm{Hur}(\mathbb{H})$. Therefore we are looking for a different identification of $\mathbb{H}$ in $\mathbb{O}$, namely the one matching the canonical basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ of $\mathbb{H}$ with (u_0, u_3, u_4, u_6). Using (9.5.5) again, it follows that

$$\text{(1)} \qquad D := \mathrm{Hur}(\mathbb{H}) = \mathbb{Z}u_0 \oplus \mathbb{Z}u_3 \oplus \mathbb{Z}u_4 \oplus \mathbb{Z}\mathbf{h}, \quad \mathbf{h} = \frac{1}{2}(u_0 + u_3 + u_4 + u_6)$$

belongs to $\mathrm{Cox}(\mathbb{O})$. Since also the element $\mathbf{p}$ of (9.5.3) belongs to $\mathrm{Cox}(\mathbb{O})$, so does $D \oplus D\mathbf{p}$, this obviously being a direct sum of abelian groups. We claim

$$\text{(2)} \qquad \mathrm{Cox}(\mathbb{O}) = D \oplus D\mathbf{p},$$

where what we have just seen and Prop. 3.1 show that the right-hand side is a unital subalgebra of the left. In order to prove equality, we apply (9.5.3) to obtain

$$u_4\mathbf{p} = \frac{1}{2}(-u_1 + u_2 + u_4 - u_6),$$

and (1) yields

$$u_1 = u_4 - \mathbf{h} + \mathbf{p} - u_4\mathbf{p} \in D \oplus D\mathbf{p}.$$

Hence $u_2 = u_4 u_1 \in D \oplus D\mathbf{p}$ as well, and with E' as in 9.5 we have shown $\mathrm{Ga}_{E'}(\mathbb{H}) \subseteq D \oplus D\mathbf{p}$. In view of (9.5.4), this completes the proof of (2).

Now one checks easily that

$$\text{(3)} \qquad D^\sharp = \big\{ \frac{1}{2} \sum \gamma_i u_i \mid \gamma_i \in \mathbb{Z}\ \ (i = 0, 3, 4, 6), \quad \sum \gamma_i \equiv 0 \bmod 2 \big\},$$

where the summations on the right are to be extended over $i = 0, 3, 4, 6$. Combining (2) with Prop. 3.5 and (3), we deduce

$$\text{(4)} \qquad \mathrm{Cox}(\mathbb{O}) = \mathrm{Cay}(D; t, -1),$$

where $t\colon D \to \mathbb{Z}$ is the linear form given by

$$(5) \qquad t(w) = n_{\mathbb{O}}(\mathbf{p}, w) = n_{D_{\mathbb{Q}}}(b, w), \quad b = \frac{1}{2}b_0 \in D^\sharp, \quad b_0 = u_0 + u_3 \qquad (w \in D)$$

since $u_1, u_2 \in D^\perp \subseteq \mathrm{Cox}(\mathbb{O})$; thus t is characterized by $t(u_0) = t(u_3) = t(\mathbf{h}) = 1$, $t(u_4) = 0$. In the notation of Thm. 8.10, we may choose $\delta = 2$ by (3) and have $\mu = -1$, $\varepsilon = 2$, hence $\varepsilon + \delta^2\mu = -2 \notin \mathbb{Z}^\times$. In particular, *the converse of the implication* 8.10 (d) *does* not *hold.* Hence, if we had started from D (resp. t) as given in (1) (resp. (5)) to define a conic algebra over the integers by the right-hand side of (4), we could not at all be sure to obtain an octonion algebra in this way, unless we were able to identify it with the Coxeter octonions.

9.7. **Example: the Dickson octonions.** Keeping the notation of the previous subsection, Dickson [3, p. 319] defined what we call the *Dickson octonions* as

$$(1) \qquad \mathrm{Dic}(\mathbb{O}) := D \oplus \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3 \oplus \mathbb{Z}v_4 \subseteq \mathbb{O},$$

where

$$(2) \qquad v_1 := u_2,$$

$$(3) \qquad v_2 := \frac{1}{2}(u_0 - u_1 + u_2 + u_3),$$

$$(4) \qquad v_3 := \frac{1}{2}(u_0 + u_2 + u_4 - u_5),$$

$$(5) \qquad v_4 := \frac{1}{2}(u_0 + u_2 + u_6 - u_7).$$

Dickson showed by direct computation that $\mathrm{Dic}(\mathbb{O})$ is an octonion algebra over the integers. Within the context of the present work, this can be accomplished much easier by proving the following result.

9.8. **Proposition.** $\mathrm{Dic}(\mathbb{O}) = \mathrm{Cox}(\mathbb{O})$.

Proof. Since $D \subseteq \mathrm{Cox}(\mathbb{O})$ by (9.6.2), the left-hand side will be contained in the right once we have shown $v_i \in \mathrm{Cox}(\mathbb{O})$ for $i = 1, 2, 3, 4$. Using (9.5.5), this follows by straightforward computations. It remains to show $\mathrm{Cox}(\mathbb{O}) \subseteq \mathrm{Dic}(\mathbb{O})$.

As a first step we prove $\mathrm{Ga}_E(\mathbb{O}) \subseteq \mathrm{Dic}(\mathbb{O})$. By (9.6.1) and (9.7.1), we clearly have $u_0, u_3, u_4, \mathbf{h} \in D \subseteq \mathrm{Dic}(\mathbb{O})$, while (9.6.1) and (9.7.1)–(9.7.5) yield

$$(1) \qquad u_2 = v_1 \in \mathrm{Dic}(\mathbb{O}),$$

$$(2) \qquad u_1 = -2v_2 + u_0 + u_2 + u_3 \in \mathrm{Dic}(\mathbb{O}),$$

$$(3) \qquad u_5 = -2v_3 + u_0 + u_2 + u_4 \in \mathrm{Dic}(\mathbb{O}),$$

$$(4) \qquad u_6 = 2\mathbf{h} - u_0 - u_3 - u_4 \in \mathrm{Dic}(\mathbb{O}),$$

$$(5) \qquad u_7 = -2v_4 + u_0 + u_2 + u_6 \in \mathrm{Dic}(\mathbb{O}),$$

hence proves our intermediate assertion.

Returning to the quantity $\mathbf{p}$ of (9.5.3), we not only have $\mathbf{p} = u_1 + v_2 \in \mathrm{Dic}(\mathbb{O})$ by (9.7.3) but also, using (1)–(5),

$$\begin{aligned} u_1\mathbf{p} &= \frac{1}{2}(-u_0 + u_1 + u_4 + u_7) \equiv \frac{1}{2}(-u_0 + u_0 + u_2 + u_3 + u_4 + u_0 + u_2 + u_6) \\ &\equiv \frac{1}{2}(u_0 + u_3 + u_4 + u_6) \equiv \mathbf{h} \equiv 0 \bmod \mathrm{Dic}(\mathbb{O}), \\ u_2\mathbf{p} &= \frac{1}{2}(-u_0 + u_2 - u_4 + u_5) \equiv \frac{1}{2}(-u_0 + u_2 + u_4 + u_0 + u_2 + u_4) \\ &\equiv u_2 \equiv 0 \bmod \mathrm{Dic}(\mathbb{O}) \end{aligned}$$

and, finally,

$$u_4\mathbf{p} = \frac{1}{2}(-u_1 + u_2 + u_4 - u_6) \equiv \frac{1}{2}(-u_0 - u_2 - u_3 + u_2 + u_4 - u_6)$$
$$\equiv \frac{1}{2}(-u_0 - u_3 + u_4 - u_6) \equiv -u_0 - u_3 - u_6 + \mathbf{h} \equiv 0 \bmod \mathrm{Dic}(\mathbb{O}).$$

Defining $B \subseteq \mathrm{Dic}(\mathbb{O})$ as in (9.5.2), we have thus shown $B\mathbf{p} \subseteq \mathrm{Dic}(\mathbb{O})$, which in view of (9.5.4) proves $\mathrm{Cox}(\mathbb{O}) \subseteq \mathrm{Dic}(\mathbb{O})$ and completes the proof. □

Remark. Combining Prop. 9.8 with (9.6.4), we obtain an explicit realization of the Dickson octonions out of the Hurwitz quaternions by means of the non-orthogonal Cayley-Dickson construction.

9.9. **The sedenions.** Physicists love the *sedenions* (see, e.g., [4, 8, 9]), i.e., the flexible conic algebra of dimension 16 over the reals defined by $\mathbb{S} := \mathrm{Cay}(\mathbb{O}, -1)$. Contrary to what one would expect from the analogy to composition algebras, the sedenions have a positive definite, hence anisotropic, norm and yet admit zero divisors; in fact, the *zero divisor pairs* ($(x, y) \in \mathbb{S}^2$ satisfying $\|x\| = \|y\| = 1$ and $xy = 0$) form a principal homogeneous space for the compact Lie group G_2 [13], [16, Exc. 113].

9.10. **Vista: integral sedenions.** According to Witt [20], there are essentially two unimodular positive definite integral quadratic lattices of rank 16 over the integers: the direct sum of two copies of the E_8-lattice, and a unique indecomposable one. Using the non-orthogonal Cayley-Dickson construction, we will now show that *at least one, and possibly both, of these may be endowed with the structure of a sedenion algebra over* $\mathbb{Z}$.

As in 9.5, let $(u_i)_{0\leq i\leq 7}$ be a Cartan-Shouten basis of $\mathbb{O}$. Then $A := \mathrm{Ga}_E(\mathbb{O}) \subseteq \mathbb{O}$ is a weakly non-singular alternative conic algebra over $\mathbb{Z}$ with positive definite norm such that $A^\sharp = \frac{1}{2}A$. Let $(\alpha_i)_{0\leq i\leq 7}$ be any family of elements in $\{0, \pm 1\}$, precisely three of which are assumed to be non-zero, and define a linear form s on A by

$$s = n_{A_{\mathbb{Q}}}\big(\frac{1}{2}\sum \alpha_i u_i, -\big)|_A \colon\ A \longrightarrow \mathbb{Z}.$$

Then Thm. 8.10 combined with Cor. 8.4 shows that

$$C := \mathrm{Cay}(A; -1, s)$$

is a nonsingular conic algebra of rank 16 with positive definite norm form over $\mathbb{Z}$. Hence its underlying integral quadratic lattice must be one of the two mentioned above. For example, if $\alpha_0 = \alpha_1 = \alpha_2 = 1$ and $\alpha_4 = \cdots = \alpha_7 = 0$, a comparison with (9.5.6) and (9.5.7) shows that C contains the Coxeter octonions as a unital subalgebra. Hence the integral quadratic lattice underlying C splits into the direct sum of two copies of the E_8-lattice. Whether an analogous example can be found where the underlying integral quadratic lattice is indecomposable remains an open question.

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