Polar Decompositions of Quaternion Algebras over arbitrary Rings

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1. Introduction

Over a commutative ring containing $\frac{1}{2}$, all quaternion algebras in the sense of Petersson [4], which are the separable quaternion algebras in the sense of Knus [1], are known since the work of Pumplün [6] to arise from ternary symmetric bilinear spaces by means of their polar decomposition. The reader is referred to subsection 3.3 below for details. Our aim in this paper will be to present a more general construction that does not require anymore the presence of $\frac{1}{2}$ in the base ring. For a quaternion algebra to arise from this construction it is necessary and sufficient that it contain invertible elements of trace 1, which it does automatically not only if $\frac{1}{2}$ exists but also over fields of characteristic two. Along more technical lines, the ternary bilinear spaces connected with our new type of polar decomposition are in general not symmetric. Our approach is reminiscent of, but actually much simpler than, the one adopted by Loos [2] for considerably more general purposes.

Throughout this paper, we let k be an arbitrary commutative associative ring of scalars.

2. Non-symmetric bilinear spaces

2.1. Our aim in this section will be to collect a few elementary observations about bilinear forms.

2.2. Bilinear modules. A *bilinear module* over k is a pair (M, f) consisting of a k-module M and a bilinear form $f : M \times M \to k$, not necessarily symmetric. A *homomorphism* $\varphi : (M, f) \to (M', f')$ of bilinear modules over k is a k-linear map $\varphi : M \to M'$ that respects the forms: $f' \circ (\varphi \times \varphi) = f$. Injective homomorphisms of bilinear modules are called *isometries.* A *bilinear submodule* of (M, f) has the from $(N, f|_{N\times N})$ for some submodule $N \subseteq M$.

2.3. Examples: Matrices. Examples of bilinear modules are $(k^n, \langle S\rangle)$ for $n \in \mathbb{N}, S \in \mathrm{Mat}_n(k)$, where the bilinear form $\langle S\rangle$ on k^n is given by

$$\langle S\rangle : k^n \times k^n \longrightarrow k\,,\ (x, y) \longmapsto \langle S\rangle(x, y) := x^t S y\,; \qquad (2.3.1)$$

up to isomorphism, they exhaust all bilinear modules over k whose underlying k-module is free of rank n.

2.4. Exterior powers and determinants. Recall that the n-th exterior power ($n \in \mathbb{N}$) of a bilinear module (M, f) over k is

$$\bigwedge^n (M, f) = (\bigwedge^n M, \bigwedge^n f)\,,$$

where $\bigwedge^n f : (\bigwedge^n M) \times (\bigwedge^n M) \to k$ is defined by

$$(\bigwedge^n f)(x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n) := \det\big(f(x_i, y_j)\big)\,. \qquad (x_i, y_j \in M, i, j \in \mathbb{N}_n) \tag{2.4.1}$$

Given a k-linear bijection $\Delta : \bigwedge^n M \xrightarrow{\sim} k$ (which may not exist, but if it does is unique up to a factor in $k^\times$), 2.3 yields a unique element $\det_\Delta f \in k$, making

$$\Delta : \bigwedge^n (M, f) = (\bigwedge^n M, \bigwedge^n f) \xrightarrow{\sim} (k, \langle \det_\Delta f \rangle)$$

a bijective isometry. By (2.3.1), (2.4.1), this amounts to

$$(\det_\Delta f)\Delta(x_1 \wedge \cdots \wedge x_n)\Delta(y_1 \wedge \cdots \wedge y_n) = \det\big(f(x_i, y_j)\big)\,. \qquad (x_i, y_j \in M, i, j \in \mathbb{N}_n) \tag{2.4.2}$$

The following observation is an immediate consequence of the definitions.

2.5. Fact. *Notations and assumptions being as in* 2.4, $\det_\Delta f$ *is bi-homogeneous of bi-degree* $(-2, n)$ *in* (Δ, f). $\square$

2.6. Determinants for a module. Let M be a finitely generated projective k-module of constant rank $n \in \mathbb{N}$. Then a *determinant* for M is a k-linear isomorphism $\bigwedge^n M \xrightarrow{\sim} k$.

2.7. Examples: Matrices revisited. Denoting by $(e_1, \ldots, e_n)$ the basis of unit vectors in k^n ($n \in \mathbb{N}$) and by Δ^n the determinant for k^n sending $e_1 \wedge \cdots \wedge e_n$ to 1, (2.3.1) and (2.4.2) yield

$$\det_{\Delta^n} \langle S \rangle = \det S\,. \qquad (S \in \mathrm{Mat}_n(k)) \tag{2.7.1}$$

2.8. Non-singularity. The dual module of a k-module M will be denoted by $M^* = \mathrm{Hom}_k(M, k)$. A bilinear module (M, f) over k is said to be *non-singular* if the linear maps

$${}_f\varphi : M \longrightarrow M^*\,,\ x \longmapsto f(x, -)\,;\ \varphi_f : M \longrightarrow M^*\,,\ y \longmapsto f(-, y)$$

are both bijective. For $n \in \mathbb{N}, S \in \mathrm{Mat}_n(k)$, we clearly have

$$(k^n, \langle S \rangle) \text{ is non-singular} \iff S \in \mathrm{GL}_n(k).$$

A(n *n-ary*) *bilinear space* over k is a bilinear module (M, f) with M finitely generated projective (of constant rank n) and f non-singular. A *bilinear subspace* is a bilinear submodule in the sense of 2.2 which is a bilinear space at the same time.

2.9. Proposition. *Let (M, f) be a bilinear module over k and suppose M is finitely generated projective. Then the following statements are equivalent.*

(i) *(M, f) is a bilinear space.*

(ii) *${}_f\varphi : M \xrightarrow{\sim} M^*$ is bijective.*

(iii) *$\varphi_f : M \xrightarrow{\sim} M^*$ is bijective.*

In this case, there exists a unique map $\theta_f : M \to M$ satisfying the relation

$$f(y, x) = f\big(\theta_f(x), y\big)\,, \qquad (x, y \in M) \tag{2.9.1}$$

and θ_f is a linear isomorphism.

Proof. The equivalence of (i), (ii), (iii) may be checked locally, allowing us to assume $(M, f) = (k^n, \langle S\rangle), n \in \mathbb{N}, S \in \mathrm{Mat}_n(k)$. Then both (ii) and (iii) are equivalent to $S \in \mathrm{GL}_n(k)$, and the assertion follows. As to the final part, the only map satisfying (2.9.1) is $\theta_f = ({}_f\varphi)^{-1} \circ (\varphi_f)$. $\square$

2.10. Ternary bilinear spaces and the associated vector product. Let (M, f) be a ternary (i.e., 3-ary) bilinear space over k and suppose we are given a determinant Δ for M. By non-singularity, there is a unique map

$$\times_{f,\Delta} : M \times M \longrightarrow M\,,\ (x, y) \longmapsto x \times_{f,\Delta} y\,,$$

called the *vector product* associated with f, Δ, such that

$$f(x \times_{f,\Delta} y, z) = \Delta(x \wedge y \wedge z)\,. \qquad (x, y, z \in M) \tag{2.10.1}$$

Notice the analogy to the hermitian vector product discussed in [5]. The vector product is clearly bilinear and alternating; it also satisfies the obvious relations

$$f(x \times_{f,\Delta} y, z) = f(y \times_{f,\Delta} z, x)\,, \qquad (x, y, z \in M) \tag{2.10.2}$$

$$f(x \times_{f,\Delta} y, z) = 0\,. \qquad (x, y, z \in M, |\{x, y, z\}| \le 2) \tag{2.10.3}$$

Finally, we observe

$$x \times_{\alpha f, \beta\Delta} y = \alpha^{-1}\beta(x \times_{f,\Delta} y)\,, \qquad (\alpha, \beta \in k^\times, x, y \in M) \tag{2.10.4}$$

which follows immediately from (2.10.1).

2.11. Examples: 3-by-3 matrices. a) We write $x \times y$ for the ordinary vector product in 3-space, which is determined by the condition

$$(x \times y)^t z = \det(x, y, z) = \Delta^3(x \wedge y \wedge z)\,, \qquad (x, y, z \in k^3) \tag{2.11.1}$$

Δ^3 being understood in the sense of 2.7. Given $Q \in \mathrm{GL}_3(k)$, we also obtain the relation

$$(Qx) \times (Qy) = (Q^\sharp)^t(x \times y)\,, \tag{2.11.2}$$

which was established, e.g., in [5, (7.10.6]. Finally, we recall the classical Grassmann identity

$$(x \times y) \times z = (z^t x)y - (z^t y)x\,. \qquad (x, y, z \in k^3) \tag{2.11.3}$$

b) Now let $S \in \mathrm{GL}_3(k)$ and put $f = \langle S \rangle, \Delta = \Delta^3$. Identifying $\mathrm{End}_k(k^3) = \mathrm{Mat}_3(k)$ canonically, we claim

$$\theta_f = (S^t)^{-1} S\,, \tag{2.11.4}$$

$$x \times_{f,\Delta} y = (S^t)^{-1}(x \times y) = (\det S)^{-1}\big((Sx) \times (Sy)\big)\,. \qquad (x, y, z \in k^3) \tag{2.11.5}$$

Indeed, for all $x, y, z \in k^3$,

$$\begin{aligned} f(y, x) &= y^t S x && \text{(by (2.3.1))} \\ &= x^t S^t y = \big((S^t)^{-1} S x\big)^t S y \\ &= f\big((S^t)^{-1} S x, y\big)\,, \end{aligned}$$

which gives (2.11.4) by (2.9.1), and

$$\begin{aligned} (x \times y)^t z &= \Delta(x \wedge y \wedge z) && \text{(by (2.11.1))} \\ &= f(x \times_{f,\Delta} y, z) && \text{(by (2.10.1))} \\ &= (x \times_{f,\Delta} y)^t S z\,, \end{aligned}$$

which implies

$$\begin{aligned} x \times_{f,\Delta} y = (S^t)^{-1}(x \times y) &= (\det S)^{-1}(S^\sharp)^t (x \times y) \\ &= (\det S)^{-1}\big((Sx) \times (Sy)\big)\,, && \text{(by (2.11.2))} \end{aligned}$$

hence (2.11.5) as well.

2.12. Proposition. *Let (M, f) be a ternary bilinear space over k and Δ a determinant for M. Then the Grassmann identity*

$$(x \times_{f,\Delta} y) \times_{f,\Delta} z = (\det_\Delta f)^{-1}[f(z, x)\theta_f(y) - f(z, y)\theta_f(x)] \tag{2.12.1}$$

holds for all $x, y, z \in M$.

Proof. By 2.5 and (2.10.4), the validity of the Grassmann identity does not depend on the choice of Δ. Being also local on k, we may assume $(M, f) = (k^3, \langle S \rangle)$ for some $S \in \mathrm{GL}_3(k)$, and $\Delta = \Delta^3$. Setting $T = S^t$ and adopting the notations of 2.11, we then obtain

$$\begin{aligned} (x \times_{f,\Delta} y) \times_{f,\Delta} z &= (\det S)^{-1}[S(x \times_{f,\Delta} y) \times Sz] && \text{(by (2.11.5))} \\ &= (\det S)^{-1}[ST^{-1}(x \times y) \times Sz] && \text{(by (2.11.5))} \\ &= (\det S)^{-1}[\big((T^{-1}S)^\sharp\big)^t (x \times y) \times Sz] && \text{(since } \det(T^{-1}S) = 1, T = S^t) \\ &= (\det S)^{-1}[\big((T^{-1}Sx) \times (T^{-1}Sy)\big) \times Sz] && \text{(by (2.11.2))} \\ &= (\det S)^{-1}[(z^t S^t T^{-1} S x)(T^{-1} S y) - (z^t S^t T^{-1} S y)(T^{-1} S x)] && \text{(by (2.11.3))} \\ &= (\det_\Delta f)^{-1}[f(z, x)\theta_f(y) - f(z, y)\theta_f(x)]\,. && \text{(by (2.7.1), (2.3.1), (2.11.4))} \end{aligned}$$

$\square$

We close this section with two easy technicalities that turn out to be useful later on.

2.13. Lemma. *Let M be a finitely generated projective k-module of constant rank 3 and Δ a determinant for M. Given an alternating bilinear form $g : M \times M \to k$, there is a unique element $v \in M$ satisfying*

$$g(x, y) = \Delta(v \wedge x \wedge y). \qquad (x, y \in M) \tag{2.13.1}$$

Proof. We write $\mathrm{Alt}(M) = \mathrm{Hom}_k(\bigwedge^2 M, k)$ for the k-module of alternating bilinear forms on M, which is finitely generated projective of constant rank $\binom{3}{2} = 3$. Clearly, the right-hand side of (2.13.1) defines an element $\Delta_v \in \mathrm{Alt}(M)$, and it suffices to show that the assignment $v \mapsto \Delta_v$ gives a k-linear bijection $M \xrightarrow{\sim} \mathrm{Alt}(M)$. This is a local question, and since M and $\mathrm{Alt}(M)$ both have the same finite rank, we may in fact assume that k is a field. But then any $0 \neq v \in M$ extends to a basis v, x, y of M, which implies $\Delta_v(x, y) = \Delta(v \wedge x \wedge y) \neq 0$, so $v \mapsto \Delta_v$ is injective, and the assertion follows. □

2.14. Lemma. *Let (M, f) be a ternary bilinear space over k and suppose k is a local ring. Then (M, f) contains a binary bilinear subspace.*

Proof. Writing κ for the residue field of k, $(M, f) \otimes \kappa$ is a ternary bilinear space over κ, which cannot be alternating. We therefore find an element $e \in M$ such that $f(e, e) \in k^\times$. Then

$$N := \{x \in M | f(e, x) = 0\} \subseteq M$$

is a free submodule of rank 2 satisfying $M = ke \oplus N$. We claim that the bilinear submodule $(N, f|_{N \times N})$ of (M, f) is non-singular. It suffices to check this over κ, so we may assume from the outset that k itself is a field. If $y \in N$ satisfies $f(N, y) = \{0\}$, we may combine this with $f(e, y) = 0$ to conclude $f(M, y) = \{0\}$, hence $y = 0$, and the proof is complete. □

3. Distinguished complements of 1 and ternary bilinear spaces.

3.1. Distinguished complements of 1 are shown in this section to form the key notion for generalizing the classical polar decomposition of quaternion algebras containing $\frac{1}{2}$, which we briefly recall in 3.3 below, to arbitrary base rings. They will also be seen to determine canonically a base point as well as a ternary bilinear space together with a determinant from which the multiplicative structure of the ambient quaternion algebra can be completely recovered.

3.2. Quaternion algebras. Throughout this section, we fix a quaternion algebra C over k in the sense of [4, 1.4 − 1.8]. Thus C is associative and finitely generated projective of constant rank 4 as a k-module, contains a unit element and carries a quadratic form $N_C : C \to k$ uniquely determined by the following properties: N_C *permits composition*, so $N_C(xy) = N_C(x)N_C(y)$ for all $x, y \in C$, and N_C is *non-singular*, so its bilinearization, also denoted by

$$N_C : C \times C \longrightarrow k\,,\ (x, y) \longmapsto N_C(x, y) = N_C(x + y) - N_C(x) - N_C(y)\,,$$

canonically determines an isomorphism from the k-module C onto its dual $C^* = \mathrm{Hom}_k(C, k)$. Calling N_C the *norm* of C, we also write $1 = 1_C$ for the unit element, $T_C = N_C(1, -) : C \to k$ for the *trace* and

$$\iota : C \longrightarrow C\,,\ x \longmapsto \iota(x) = \overline{x} = T_C(x)1 - x\,, \tag{3.2.1}$$

for the *conjugation* of C, which is an algebra involution in the usual sense; it relates to the norm by the formula

$$N_C(y, \overline{x}\, z) = N_C(xy, z) = N_C(x, z\, \overline{y}\,). \qquad (x, y, z \in C) \tag{3.2.2}$$

Other standard properties of quaternions will be used here without further comment. We only mention explicitly that quaternion algebras are *unitally faithful* in the sense of McCrimmon [3, p. 85], so the map $\alpha \mapsto \alpha 1$ from k to C is injective. Indeed, we are allowed to check this locally and hence may assume that k is a local ring, with maximal ideal $\mathfrak{m}$. But then $1_C \notin \mathfrak{m}C$ extends to a basis of C, and the assertion follows.

3.3. The standard polar decomposition. For the time being, let us assume $\frac{1}{2} \in k$. Then we write $M = C^0 = \ker T_C \subseteq C$ for the k-submodule of *pure* quaternions and obtain the orthogonal splitting

$$C = k1 \oplus M \tag{3.3.1}$$

relative to N_C. Hence, setting $f := \frac{1}{2} N_C|_{M \times M} : M \times M \to k$, (M, f) is a ternary symmetric bilinear space over k, and following [6, Proposition 2.7], there is a unique determinant Δ for M such that the multiplication of C may be recovered from f and the vector product associated with f, Δ by the formula

$$(\alpha 1 + x)(\beta 1 + y) = \big(\alpha\beta - f(x, y)\big)1 + (\alpha y + \beta x + x \times_{f,\Delta} y) \qquad (\alpha, \beta \in k, x, y \in M) \tag{3.3.2}$$

We call (3.3.1) together with the rule (3.3.2) the *standard polar decomposition* of C. It is canonical in the sense that it will be preserved by isomorphisms of quaternion algebras.

3.4. Complements of 1. Returning to the case of an arbitrary base ring, we continue to work with a quaternion algebra C over k. By a *complement of* 1 in C we mean a submodule $M \subseteq C$ that is complementary to 1:

$$C = k1 \oplus M\,. \tag{3.4.1}$$

Notice, however, that this decomposition can never be orthogonal relative to N_C unless $\frac{1}{2} \in k$. On the other hand, C being unitally faithful by 3.2, we always obtain unique linear maps $\lambda_M : C \to k$,

$$\pi_M : C \longrightarrow M$$

such that $x = \lambda_M(x)1 + \pi_M(x)$ $(x \in C)$. But N_C is non-singular, so there is a unique element $e_M \in C$ satisfying $\lambda_M = N_C(e_M, -)$, and we end up with the relation

$$x = N_C(e_M, x)1 + \pi_M(x)\,. \qquad (x \in C) \tag{3.4.2}$$

Comparing with (3.4.1), we conclude that $e_M \in C$ has trace 1 and $M = e_M^{\perp}$. Conversely, let $e \in C$ have trace 1 and put $M_e = e^{\perp}$. Then (3.4.1) holds, so M_e is a complement of 1 in C, and $\lambda_{M_e} = N_C(e, -)$. Summing up, we have established $M \mapsto e_M, e \mapsto M_e$ as inverse bijections between the complements of 1 in C and the elements of C having trace 1.

3.5. Polar decompositions. Let M be a complement of 1 in C. We wish to describe the algebra structure of C in terms of the decomposition (3.4.1). To this end, we observe that M is finitely generated projective of constant rank 3 and let $e = e_M \in C$ be the element of trace 1 corresponding to M by 3.4. We then define bilinear maps

$$f_M : M \times M \longrightarrow k\,,\ \times_M : M \times M \longrightarrow M$$

by

$$f_M(x,y) := -N_C(e,xy)\,,\ x \times_M y := \pi_M(xy) - T_M(y)x\,. \qquad (x,y \in M, T_M := T_C|_M) \tag{3.5.1}$$

Combining (3.5.1) with (3.4.2) for xy in place of x, we conclude

$$\begin{aligned}(\alpha 1 + x)(\beta 1 + y) = \big(\alpha\beta - f_M(x,y)\big)1 + &\\ (\alpha y + [\beta + T_M(y)]x + x \times_M y)\,. &\qquad (\alpha,\beta \in k, x,y \in M)\end{aligned} \tag{3.5.2}$$

We call (3.4.1) together with the rule (3.5.2) the *polar decomposition of C relative to M*. It differs from the standard polar decomposition 3.3 by the additive term $T_M(y)x$ in the multiplication rule and is no longer canonical since it is not preserved in general by isomorphisms of quaternion algebras. On the positive side, (3.5.1) combines with (3.2.2) to yield

$$f_M(x,y) = -N_C(\overline{x}\,e,y) = -N_C(e\,\overline{y}\,,x)\,. \qquad (x,y \in M) \tag{3.5.3}$$

Also, setting $\alpha = \beta = 0, x = y$ in (3.5.2), we obtain

$$x^2 = -f_M(x,x)1 + T_C(x)x + x \times_M x\,,$$

while we always have $x^2 = T_C(x)x - N_C(x)1$. Hence

$$f_M(x,x) = N_C(x)\,,\ x \times_M x = 0\,. \qquad (x \in M) \tag{3.5.4}$$

3.6. Distinguished complements of 1 and base points. Let $M \subseteq C$ be a complement of 1 in C. We say that M is *distinguished* if $e = e_M$ is an invertible element of C. In this case, we put

$$v := v_M := 2 \cdot 1_C - N_C(e)^{-1}e \tag{3.6.1}$$

and obtain $N_C(e,v) = 2T_C(e) - N_C(e)^{-1}N_C(e,e) = 0$, so v belongs to $e^\perp = M$. We call v the *base point* of M. Hence

$$e = 2N_C(e)1_C - N_C(e)v$$

is the representation of e in the polar decomposition of C relative to M. The equation

$$\begin{aligned} f_M(v,v) &= N_C(v) && \text{(by (3.5.4))}\\ &= N_C\big(2 \cdot 1_C - N_C(e)^{-1}e\big) && \text{(by (3.6.1))}\\ &= 4 - 2N_C(e)^{-1}T_C(e) + N_C(e)^{-2}N_C(e) &&\\ &= 4 - N_C(e)^{-1} && \end{aligned}$$

implies

$$4 - f_M(v,v) = N_C(e)^{-1} \in k^\times\,. \tag{3.6.2}$$

3.7. Example. For $\frac{1}{2} \in k$, the pure quaternions $M = C^0 \subseteq C$ as in 3.3 form a distinguished complement of 1 (since $e_M = \frac{1}{2}1_C \in C^\times$) whose base point is $v_M = 0$, by (3.6.1).

3.8. Proposition. a) *A complement M of* 1 *in C is distinguished if and only if (M, f_M) is a ternary bilinear space.*
b) *For distinguished complements of* 1 *in C to exist it is necessary and sufficient that C contain invertible elements of trace* 1.

Proof. a) Assume first that M is distinguished, so $e = e_M \in C^\times$. The question of f_M being non-singular is local on k and we may in fact suppose that k is a field. If $x \in M$ satisfies $f_M(x, y) = 0$ for all $y \in M$, then (3.5.3) implies $N_C(\overline{x}\,e, M) = \{0\}$, while we always have $N_C(\overline{x}\,e, 1_C) = N_C(e, x) = 0$ since $M = e^\perp$. Thus $N_C(\overline{x}\,e, C) = \{0\}$, which implies $x = 0$ since e is invertible, forcing (M, f_M) to be a ternary bilinear space. Conversely, let this be so and assume that e is not invertible in C. Then some $\mathfrak{p} \in \operatorname{Spec} R$ contains $N_C(e)$, and after changing scalars to $\kappa(\mathfrak{p})$ we are reduced to the case $N_C(e) = 0$. But then $e \in e^\perp = M$ and, for all $y \in M$,

$$\begin{aligned} f_M(e, y) &= -N_C(\overline{e}\,e, y) && \text{(by (3.5.3))} \\ &= -N_C(e)T_C(y) = 0\,, \end{aligned}$$

a contradiction. This completes the proof of a).
b) is an immediate consequence of the definition combined with 3.4. ☐

3.9. Proposition. *Let M be a distinguished complement of* 1 *in C. Then there exists a unique determinant Δ_M for M such that $\times_M = \times_{f_M, \Delta_M}$ is the vector product associated with f_M, Δ_M:*

$$f_M(x \times_M y, z) = \Delta_M(x \wedge y \wedge z)\,. \qquad (x, y, z \in M) \tag{3.9.1}$$

Furthermore, writing $v = v_M$ for the base point of M, the relations

$$f_M(v, x) = T_M(x) = f_M(x, v)\,, \tag{3.9.2}$$

$$f_M(x, y) - f_M(y, x) = T_M(x \times_M y) = \Delta_M(v \wedge x \wedge y) \tag{3.9.3}$$

hold for all $x, y \in M$.

Proof. C being an associative algebra, we may compute mod M and compare

$$xy^2 = T_C(y)xy - N_C(y)x \equiv -T_C(y)f_M(x, y)1 \bmod M \qquad \text{(by (3.5.2))}$$

with

$$\begin{aligned} (xy)y &\equiv T_M(y)xy + (x \times_M y)y \bmod M \\ &\equiv -[T_C(y)f_M(x, y) + f_M(x \times_M y, y)]1 \bmod M \end{aligned}$$

to conclude $f_M(x \times_M y, y) = 0$ from unital faithfulness. In conjunction with (3.5.4), this shows that the 3-linear expression $f_M(x \times_M y, z)$ is alternating in $x, y, z \in M$. Hence there exists a unique linear map $\Delta = \Delta_M : \bigwedge^3 M \to k$ satisfying (3.9.1). We claim that Δ is bijective. This assertion being local on k, we may assume that k is a local ring and, after reducing modulo its maximal ideal, even that k is a field. Then our assertion reduces to proving $\Delta \neq 0$. Assume the contrary. Since f_M is non-singular by Proposition 3.8 a), this implies $x \times_M y = 0$ for all $x, y \in M$, and (3.5.2) for $\alpha = 0$ reads

$$x(\beta 1 + y) = -f_M(x, y)1 + [\beta + T_M(y)]x\,,$$

so the left multiplication by any $x \in M$ has rank at most 2. Thus M, consisting entirely of non-invertible elements, must be a three-dimensional totally isotropic space relative to N_C. This contradiction proves our assertion, and it remains to establish (3.9.2), (3.9.3). To this end, we compute

$$\begin{aligned} f_M(v,x) &= -N_C(\overline{v}\,e,x) && \text{(by (3.5.3))} \\ &= -2N_C(e,x) + N_C(e)^{-1}N_C(\overline{e}\,e,x) && \text{(by (3.6.1))} \\ &= T_C(x) && \text{(since } M = e^{\perp} \text{ by 3.4)} \\ &= T_M(x)\,, && \text{(by (3.5.1))} \end{aligned}$$

giving the first equation of (3.9.2), while the second one follows analogously. To establish (3.9.3), we again compute mod M and compare

$$\begin{aligned} \overline{xy} &= \overline{-f_M(x,y)1 + [T_M(y)x + x \times_M y]} && \text{(by (3.5.2))} \\ &\equiv \big(-f_M(x,y) + T_M(x)T_M(y) + T_M(x \times_M y)\big)1 \bmod M && \text{(by (3.2.1))} \end{aligned}$$

with

$$\begin{aligned} \overline{y}\,\overline{x} &= \big(T_C(y)1 - y\big)\big(T_C(x)1 - x\big) && \text{(by (3.2.1))} \\ &\equiv T_M(x)T_M(y)1 + yx \bmod M \\ &\equiv \big(-f_M(y,x) + T_M(x)T_M(y)\big)1 \bmod M\,. \end{aligned}$$

The conjugation being an algebra involution, this yields the first equation of (3.9.3). The second one follows from (3.9.1), (3.9.2) since

$$T_M(x \times_M y) = f_M(x \times_M y, v) = \Delta(x \wedge y \wedge v) = \Delta(v \wedge x \wedge y)\,.$$

□

4. Ternary bilinear spaces and algebras of degree two.

4.1. We now reverse the point of view adopted in the previous section by starting from a ternary bilinear space with determinant to construct a k-algebra of degree two in the sense of McCrimmon [3]. We then proceed to derive conditions that are necessary and sufficient for this algebra to be quaternion. Combined with the results obtained before, this will complete the characterization of those quaternion algebras that allow a polar decomposition in the sense of 3.5.

4.2. Algebras of degree two. Following the terminology of McCrimmon [3, p. 86] for the concepts introduced in [4, 1.1], a non-associative algebra C over k is said to be *of degree two* if it is finitely-generated projective as a k-module, contains a unit element, and admits a quadratic form $N_C : C \to k$ satisfying $N_C(1_C) = 1$ and

$$x^2 - T_C(x)x + N_C(x)1_C = 0 \qquad (T_C = N_C(1_C, -))$$

for all $x \in C$. By [4, Lemma 1.1], N_C is unique, allowing us to call N_C (resp. T_C) the *norm* (resp. *trace*) of C, and as in (3.2.1), we have the *conjugation*

(4.2.1) $$\iota : C \longrightarrow C\,,\; x \longmapsto \iota(x) = \overline{x} = T_C(x)1_C - x\,.$$

Contrary to 3.2, however, ι need not be an algebra involution of C.

4.3. The basic construction. Let (M, f) be a ternary bilinear space over k and Δ a determinant for M. The expression $f(x, y) - f(y, x)$ being alternating bilinear in $x, y \in M$, we may play the game of (3.9.3) and apply Lemma 2.13 to obtain a unique element

$$v = v(M, f, \Delta) \in M$$

satisfying

$$f(x, y) - f(y, x) = \Delta(v \wedge x \wedge y)\,, \qquad (x, y \in M) \tag{4.3.1}$$

which in addition gives rise to a linear form

$$T : M \longrightarrow k\,,\ x \longmapsto T(x) := f(x, v) = f(v, x) \tag{4.3.2}$$

as in (3.9.2). Taking (3.5.2) as a guide, we now use the vector product associated with f, Δ (cf. 2.10) to define a non-associative algebra structure on the k-module

$$k \oplus M = \{\alpha \oplus x | \alpha \in k, x \in M\}$$

by the multiplication

$$(\alpha \oplus x)(\beta \oplus y) := \big(\alpha\beta - f(x, y)\big) \oplus \big(\alpha y + [\beta + T(y)]x + x \times_{f,\Delta} y\big)\,. \quad (\alpha, \beta \in k, x, y \in M) \tag{4.3.3}$$

The resulting k-algebra will be denoted by $C(M, f, \Delta)$. Clearly, $C(M, f, \Delta)$ is finitely generated projective of constant rank 4 as a k-module. Whenever convenient, we will identify $M \subseteq C(M, f, \Delta)$ as a submodule through the second summand.

4.4. Proposition. *Notations and assumptions being as in* 4.3, $C = C(M, f, \Delta)$ *is a* k*-algebra of degree two, with unit element, norm, polarized norm, trace, conjugation given by the formulae*

$$1_C = 1 \oplus 0\,, \tag{4.4.1}$$

$$N_C(\alpha \oplus x) = \alpha^2 + \alpha T(x) + f(x, x)\,, \tag{4.4.2}$$

$$N_C(\alpha \oplus x, \beta \oplus y) = 2\alpha\beta + \alpha T(y) + \beta T(x) + f(x, y) + f(y, x)\,, \tag{4.4.3}$$

$$T_C(\alpha \oplus x) = 2\alpha + T(x)\,, \tag{4.4.4}$$

$$\iota(\alpha \oplus x) = \overline{\alpha \oplus x} = \big(\alpha + T(x)\big) \oplus (-x) \tag{4.4.5}$$

for all $\alpha, \beta \in k, x, y \in C$.

Proof. An inspection of (4.3.3) shows that $1_C = 1 \oplus 0$ is the unit element of C. Similarly, setting $\alpha = \beta, x = y$ in (4.3.3) and observing that the vector product is alternating, we obtain

$$\begin{aligned}(\alpha \oplus x)^2 &= \big(\alpha^2 - f(x, x)\big) \oplus \big(2\alpha + T(x)\big)x \\ &= \big(2\alpha + T(x)\big)(\alpha \oplus x) - \big(\alpha^2 + \alpha T(x) + f(x, x)\big)(1 \oplus 0)\,.\end{aligned}$$

Hence (4.4.2) – (4.4.4) hold, while (4.4.5) follows immediately from (4.2.1). □

4.5. Example. Let C be a quaternion algebra over k and $M \subseteq C$ a distinguished complement of 1. Defining f_M by (3.5.1), Δ_M by Proposition 3.9, we may combine (3.5.2) with Propositions 3.8, 3.9 to find a canonical isomorphism

$$C \cong C(M, f_M, \Delta_M)\,.$$

4.6. Some useful identities. Notations and assumptions being as in 4.3, we wish to derive conditions that are necessary and sufficient for $C(M, f, \Delta)$ to be a quaternion algebra. To this end, we need to understand more fully the two parameters $\det_\Delta f, \theta_f$ involved in the Grassmann identity (2.12.1) and begin by deriving a number of useful identities. Setting

$$\delta := (\det_\Delta f)^{-1} \in k^\times ,$$

we claim, for all $x, y \in M$,

$$f(x,y) - f(y,x) = T(x \times_{f,\Delta} y) , \tag{4.6.1}$$

$$\theta_f(x) = x + x \times_{f,\Delta} v , \tag{4.6.2}$$

$$f(x, x \times_{f,\Delta} y) = \delta[T(x)f(x,y) - T(y)f(x,x)] , \tag{4.6.3}$$

$$f(x \times_{f,\Delta} y, x \times_{f,\Delta} y) = \delta[f(x,x)f(y,y) - f(x,y)f(y,x)] . \tag{4.6.4}$$

Since

$$\begin{aligned} T(x \times_{f,\Delta} y) &= f(x \times_{f,\Delta} y, v) && \text{(by (4.3.2))} \\ &= \Delta(x \wedge y \wedge v) && \text{(by (2.10.1))} \\ &= \Delta(v \wedge x \wedge y) , \end{aligned}$$

(4.6.1) follows from (4.3.1). But now

$$\begin{aligned} f(y,x) &= f(x,y) + T(y \times_{f,\Delta} x) && \text{(by (4.6.1))} \\ &= f(x,y) + f(y \times_{f,\Delta} x, v) && \text{(by (4.3.2))} \\ &= f(x,y) + f(x \times_{f,\Delta} v, y) && \text{(by (2.10.2))} \\ &= f(x + x \times_{f,\Delta} v, y) , \end{aligned}$$

which combines with (2.9.1) to yield (4.6.2). Furthermore, applying (2.10.3), we obtain

$$\begin{aligned} f(x, x \times_{f,\Delta} y) &= f(x, x \times_{f,\Delta} y) - f(x \times_{f,\Delta} y, x) \\ &= T\big(x \times_{f,\Delta} (x \times_{f,\Delta} y)\big) && \text{(by (4.6.1))} \\ &= f\big((y \times_{f,\Delta} x) \times_{f,\Delta} x, v\big) && \text{(by (4.3.2))} \\ &= f\big(\delta[f(x,y)\theta_f(x) - f(x,x)\theta_f(y)], v\big) && \text{(by (2.12.1))} \\ &= \delta[f(x,y)f(v,x) - f(x,x)f(v,y)] && \text{(by (2.9.1))} \\ &= \delta[T(x)f(x,y) - T(y)f(x,x)] , \end{aligned}$$

and this is (4.6.3). Similarly,

$$\begin{aligned} f(x \times_{f,\Delta} y, x \times_{f,\Delta} y) &= f\big(y \times_{f,\Delta} (x \times_{f,\Delta} y), x\big) && \text{(by (2.10.2))} \\ &= f\big((y \times_{f,\Delta} x) \times_{f,\Delta} y, x\big) \\ &= \delta f\big(f(y,y)\theta_f(x) - f(y,x)\theta_f(y), x\big) \\ &= \delta[f(y,y)f(x,x) - f(y,x)f(x,y)] , \end{aligned}$$

giving (4.6.4), as claimed. □

4.7. Proposition. *Let notations and assumptions be as in* 4.3.
a) *If* $\det_\Delta f = 1$, *then* N_C *permits composition*:

$$N_C\big((\alpha \oplus x)(\beta \oplus y)\big) = N_C(\alpha \oplus x)N_C(\beta \oplus y)\,. \qquad (\alpha, \beta \in k, x, y \in M) \tag{4.7.1}$$

b) *If* N_C *permits composition on* M, *so*

$$N_C(xy) = N_C(x)N_C(y)\,, \qquad (x, y \in M) \tag{4.7.2}$$

and C *is flexible* mod M, *so*

$$(xy)x \equiv x(yx) \bmod M\,, \qquad (x, y \in M) \tag{4.7.3}$$

then $\det_\Delta f = 1$.

Proof. As in 4.6, we put $\delta = (\det_\Delta f)^{-1}$. Given $x, y \in M$, we obtain

$$\begin{aligned}
N_C(xy) &= N_C\big([-f(x,y)] \oplus [T(y)x + x \times_{f,\Delta} y]\big) && \text{(by (4.3.3))}\\
&= f(x,y)^2 - T(x)T(y)f(x,y) - T(x \times_{f,\Delta} y)f(x,y) + \\
&\quad T(y)^2 f(x,x) + T(y)f(x, x \times_{f,\Delta} y) + \\
&\quad T(y)f(x \times_{f,\Delta} y, x) + f(x \times_{f,\Delta} y, x \times_{f,\Delta} y) && \text{(by (4.4.2))}\\
&= f(x,y)^2 - T(x)T(y)f(x,y) - T(x \times_{f,\Delta} y)f(x,y) + \\
&\quad T(y)^2 f(x,x) + \delta[T(x)T(y)f(x,y) - T(y)^2 f(x,x) + \\
&\quad f(x,x)f(y,y) - f(x,y)f(y,x)] && \text{(by (2.10.3) (4.6.3), (4.6.4))}\\
&= \delta f(x,x)f(y,y) + \\
&\quad (1-\delta)[f(x,y)f(y,x) - T(x)T(y)f(x,y) + T(y)^2 f(x,x)]\,. && \text{(by (4.6.1))}
\end{aligned}$$

Observing the relation $N_C(x)N_C(y) = f(x,x)f(y,y)$ by (4.4.2), we thus conclude

$$\begin{aligned}
&N_C(x)N_C(y) - N_C(xy) = \\
&\qquad (1-\delta)[f(x,x)f(y,y) - f(x,y)f(y,x) + T(y)\big(T(x)f(x,y) - T(y)f(x,x)\big)]
\end{aligned} \tag{4.7.4}$$

for all $x, y \in M$. We can now prove a). Comparing

$$\begin{aligned}
N_C\big((\alpha \oplus x)(\beta \oplus y)\big) &= N_C(\alpha\beta 1 + \alpha y + \beta x + xy)\\
&= \alpha^2\beta^2 + \alpha^2\beta T_C(y) + \alpha\beta^2 T_C(x) + \alpha^2 N_C(y) + \\
&\quad \alpha\beta[T_C(xy) + N_C(x,y)] + \beta^2 N_C(x) + \\
&\quad \alpha N_C(y, xy) + \beta N_C(x, xy) + N_C(xy)
\end{aligned}$$

with

$$\begin{aligned}
N_C(\alpha \oplus x)N_C(\beta \oplus y) &= \big(\alpha^2 + \alpha T_C(x) + N_C(x)\big)\big(\beta^2 + \beta T_C(y) + N_C(y)\big)\\
&= \alpha^2\beta^2 + \alpha^2\beta T_C(y) + \alpha\beta^2 T_C(x) + \\
&\quad \alpha^2 N_C(y) + \alpha\beta T_C(x)T_C(y) + \beta^2 N_C(x) + \\
&\quad \alpha T_C(x)N_C(y) + \beta N_C(x)T_C(y) + N_C(x)N_C(y)\,,
\end{aligned}$$

we see that, in order to establish (4.7.1), it suffices to prove the relations

$$T_C(xy) + N_C(x, y) = T_C(x)T_C(y)\,, \tag{4.7.5}$$

$$N_C(y, xy) = T_C(x)N_C(y)\,, \tag{4.7.6}$$

$$N_C(x, xy) = T_C(y)N_C(x)\,, \tag{4.7.7}$$

$$N_C(xy) = N_C(x)N_C(y) \tag{4.7.8}$$

for all $x, y \in M$. Since $\delta = 1$, (4.7.8) follows immediately from (4.7.4). For the remaining relations, we compute

$$\begin{aligned}
&T_C(xy) + N_C(x, y) \\
&\quad = T_C\big([-f(x, y)] \oplus [T(y)x + x \times_{f,\Delta} y]\big) + f(x, y) + f(y, x) && \text{(by (4.3.3), (4.4.3))} \\
&\quad = -2f(x, y) + T(x)T(y) + T(x \times_{f,\Delta} y) + f(x, y) + f(y, x) && \text{(by (4.4.4))} \\
&\quad = T_C(x)T_C(y) + f(y, x) - f(x, y) + f(x, y) - f(y, x) && \text{(by (4.6.1))} \\
&\quad = T_C(x)T_C(y)\,,
\end{aligned}$$

which gives (4.7.5),

$$\begin{aligned}
N_C(y, xy) &= N_C\big(y, [-f(x, y)] \oplus [T(y)x + x \times_{f,\Delta} y]\big) && \text{(by (4.3.3))} \\
&= -T_C(y)f(x, y) + T(y)f(y, x) + T(y)f(x, y) + \\
&\quad\; f(y, x \times_{f,\Delta} y) + f(x \times_{f,\Delta} y, y) && \text{(by (4.4.3))} \\
&= T(y)f(y, x) - T(y)f(y, x) + T(x)f(y, y) && \text{(by (4.6.3), (2.10.3))} \\
&= T_C(x)N_C(y)\,,
\end{aligned}$$

which gives (4.7.6), and, similarly,

$$\begin{aligned}
N_C(x, xy) &= N_C\big(x, [-f(x, y)] \oplus [T(y)x + x \times_{f,\Delta} y]\big) \\
&= -T_C(x)f(x, y) + 2T(y)N_C(x) + f(x, x \times_{f,\Delta} y) + f(x \times_{f,\Delta} y, x) \\
&= -T(x)f(x, y) + 2T(y)N_C(x) + T(x)f(x, y) - T(y)f(x, x) \\
&= T_C(y)N_C(x)\,,
\end{aligned}$$

which gives (4.7.7) and completes the proof of a).
To establish b), we first note that (4.7.2) and (4.7.4) imply

$$(1 - \delta)[f(x, x)f(y, y) - f(x, y)f(y, x) + T(y)\big(T(x)f(x, y) - T(y)f(x, x)\big)] = 0\,. \tag{4.7.9}$$

On the other hand,

$$\begin{aligned}
(xy)x &= -f(x, y)x + T(y)x^2 + (x \times_{f,\Delta} y)x && \text{(by (4.3.3))} \\
&\equiv -[T(y)f(x, x) + f(x \times_{f,\Delta} y, x)] \bmod M \\
&\equiv -T(y)f(x, x) \bmod M\,, && \text{(by (2.10.3))}
\end{aligned}$$

whereas

$$\begin{aligned}
x(yx) &= -f(y, x)x + T(x)xy + x(y \times_{f,\Delta} x) \\
&\equiv -[T(x)f(x, y) - f(x, x \times_{f,\Delta} y)] \bmod M \\
&\equiv -[T(x)f(x, y) - \delta T(x)f(x, y) + \delta T(y)f(x, x)] \bmod M\,, && \text{(by (4.6.3))}
\end{aligned}$$

and (4.7.3) amounts to

$$(1-\delta)\big(T(x)f(x,y)-T(y)f(x,x)\big)=0\,. \qquad (x,y\in M) \tag{4.7.10}$$

By (4.7.10), the factor of $T(y)$ in the bracket on the left of (4.7.9) vanishes, forcing

$$(1-\delta)[f(x,x)f(y,y)-f(x,y)f(y,x)]=0\,. \qquad (x,y\in M) \tag{4.7.11}$$

The assertion $\delta=1$ being local on k, we may assume that k is a local ring. But then, thanks to Lemma 2.14, (M,f) contains a binary bilinear subspace, with basis vectors x,y, say, and since the bracket on the left of (4.7.11) is the determinant of f restricted to that subspace, it must be a unit in k. This implies $\delta=1$, as desired. $\square$.

4.8. Proposition. *Let (M,f) be a ternary bilinear space over k and Δ a determinant for M. Setting $C=C(M,f,\Delta), v=v(M,f,\Delta)$ as in 4.3 and assuming*

$$4(\det_\Delta f)-f(v,v)\in k^\times\,, \tag{4.8.1}$$

N_C is a non-singular quadratic form on all of C; if, in addition, $\frac{1}{2}\in k$, then N_C is non-singular on M.

Proof. Since the hypothesis (4.8.1) is stable under base change, the assertion is not only local on k, but we may in fact assume that k is a field. Let $\alpha\oplus x$ $(\alpha\in k, x\in M)$ be an arbitrary element of C.
a) By (4.4.3), $\alpha\oplus x$ is orthogonal to 1_C (relative to N_C) if and only if

$$2\alpha+T(x)=0\,; \tag{4.8.2}$$

b) Again by (4.4.3), $\alpha\oplus x$ is orthogonal to M if and only if, for all $y\in M$,

$$\begin{aligned} 0 &= \alpha T(y)+f(x,y)+f(y,x) \\ &= f\big(\alpha v+x+\theta_f(x),y\big) && \text{(by (2.9.1), (4.3.2))} \\ &= f\big(\alpha v+2x+x\times_{f,\Delta}v,y\big)\,, && \text{(by (4.6.2))} \end{aligned}$$

i.e., if and only if

$$x\times_{f,\Delta}v=-\alpha v-2x\,. \tag{4.8.3}$$

Since

$$\begin{aligned} T(x\times_{f,\Delta}v) &= f(x,v)-f(v,x) && \text{(by (4.6.1))} \\ &= 0\,, && \text{(by (4.3.2))} \end{aligned}$$

this implies

$$\alpha T(v)=-2T(x)\,. \tag{4.8.4}$$

On the other hand, (4.8.3) also implies

$$(x\times_{f,\Delta}v)\times_{f,\Delta}v=-2(x\times_{f,\Delta}v)=2\alpha v+4x\,,$$

while (2.12.1) gives, setting $\delta = (\det_\Delta f)^{-1}$,

$$\begin{aligned}
(x \times_{f,\Delta} v) \times_{f,\Delta} v &= \delta[f(v,x)\theta_f(v) - f(v,v)\theta_f(x)] \\
&= \delta[T(x)v - T(v)x - T(v)(x \times_{f,\Delta} v)] && \text{(by (4.3.2), (4.6.2))} \\
&= \delta[T(x)v - T(v)x + \alpha T(v)v + 2T(v)x] && \text{(by (4.8.3))} \\
&= \delta[T(v)x + T(x)v - 2T(x)v] && \text{(by (4.8.4))} \\
&= \delta[T(v)x - T(x)v]\,.
\end{aligned}$$

Comparing these expressions, we conclude

$$\big(4 - \delta T(v)\big)x = -\big(2\alpha + T(x)\big)v\,. \tag{4.8.5}$$

c) We can now prove the first part of the proposition. If $\alpha \oplus x$ is orthogonal to all of C, then a),b) yield (4.8.2), (4.8.5), which combine to imply $(4 - \delta T(v))x = 0$, hence $x = 0$ by (4.8.1). But then $\alpha v = 0$ by (4.8.3) and $2\alpha = 0$ by (4.8.2), forcing

$$\big(4(\det_\Delta f) - f(v,v)\big)\alpha = 4\alpha(\det_\Delta f) - T(\alpha v) = 0\,,$$

hence $\alpha = 0$, and we have shown that N_C is indeed non-singular.
d) Finally, assume $\frac{1}{2} \in k$. If $x \in M$ is orthogonal to all of M, then b) shows that (4.8.4), (4.8.5) hold with $\alpha = 0$, which again implies $x = 0$, and the proof is complete. $\square$

We can now establish the main result of the paper.

4.9. Theorem. *Let (M,f) be a ternary bilinear space over k and Δ a determinant for M. Setting $v = v(M,f,\Delta)$ as in 4.3, $C = C(M,f,\Delta)$ is a quaternion algebra over k if and only if $\det_\Delta f = 1$ and $4 - f(v,v)$ is a unit in k. In this case, $M \subseteq C$ is a distinguished complement of 1 with base point v.*

Proof. Assume first that C is a quaternion algebra. Then Proposition 4.7 b) implies $\det_\Delta f = 1$, and M is a complement of 1 in C. Also, by (4.4.4), T is the restriction of T_C to M, which implies $T = T_M$ in the sense of (3.5.1). Comparing now (3.5.2) with (4.3.3), we conclude $f = f_M, \times_{f,\Delta} = \times_M$. In particular, M must be distinguished by Proposition 3.8 a), and (3.9.1) yields $\Delta_M = \Delta$. But then v is the base point of M, by (3.9.3), (4.3.1) and Lemma 2.13. Now (3.6.2) yields $4 - f(v,v) \in k^\times$. Conversely, suppose $\det_\Delta f = 1$ and $4 - f(v,v) \in k^\times$. Then N_C permits composition by Proposition 4.7 a) and is non-singular by Proposition 4.8, forcing C to be a quaternion algebra. $\square$

4.10. Corollary. *Assume $\frac{1}{2} \in k$, let C be a quaternion algebra over k and suppose $M \subseteq C$ is a distinguished complement of 1. Then N_C is non-singular on M.*

Proof. This follows immediately from Example 4.5, Proposition 4.8 and Theorem 4.9. $\square$

4.11. Polar decompositions and quadratic étale subalgebras. Given M, f, Δ as in 4.3 and abbreviating $v = v(M, f, \Delta)$, the relation

$$
\begin{aligned}
\det \begin{pmatrix} N_C(1,1) & N_C(1,v) \\ N_C(v,1) & N_C(v,v) \end{pmatrix} &= \det \begin{pmatrix} 2N_C(1) & T_C(v) \\ T_C(v) & 2N_C(v) \end{pmatrix} \\
&= \det \begin{pmatrix} 2 & T(v) \\ T(v) & 2f(v,v) \end{pmatrix} && \text{(by (4.4.2),(4.4.4))} \\
&= 4f(v,v) - f(v,v)^2 && \text{(by (4.3.2))} \\
&= f(v,v)\big(4 - f(v,v)\big)
\end{aligned}
$$

shows that $k[v]$, the unital subalgebra of $C = C(M, f, \Delta)$ generated by v, is quadratic étale if and only if $f(v,v)$ and $4 - f(v,v)$ are both units in k. Hence, for C to become a quaternion algebra *not* arising from the generalized Cayley-Dickson doubling process [4, 2.5], it will be necessary that $4 - f(v,v)$ is a unit in k but $f(v,v)$ is not. Trivial instances for this kind of situation are discussed in the following example.

4.12. Example. a) Consider the ternary bilinear space $(k^3, \langle S\rangle)$, $S \in \mathrm{GL}_3(k)$, and put $v = v(k^3, \langle S\rangle, \Delta^3)$ in the sense of 2.7 and 4.3. Adopting the notations of 2.11 as well, we obtain

$$(S - S^t)x = x \times v \qquad (x \in k^3) \tag{4.12.1}$$

by comparing

$$
\begin{aligned}
\langle S\rangle(x,y) - \langle S\rangle(y,x) = x^tSy - y^tSx &= x^tSy - x^tS^ty \\
&= x^t(S - S^t)y = [(S^t - S)x]^ty
\end{aligned}
$$

with

$$\Delta^3(v \wedge x \wedge y) = \det(v, x, y) = (v \times x)^t y \qquad \text{(by (2.11.1))}$$

for all $x, y \in k^3$.
b) It should be obvious that every bilinear space (V, f) over a field can be *triangularized* in the sense that there is a basis of V with respect to which the matrix of f has upper triangular form.

With this in mind, we return to our arbitrary base ring k and specialize S in a) to

$$S = \begin{pmatrix} \xi_1 & \alpha & \beta \\ 0 & \xi_2 & \gamma \\ 0 & 0 & \xi_3 \end{pmatrix}, \tag{4.12.2}$$

where $\xi_1, \xi_2, \xi_3, \alpha, \beta, \gamma \in k$ satisfy $\xi_1\xi_2\xi_3 = 1$. Then $\det_{\Delta^3}\langle S\rangle = 1$ by (2.7.1), and we have to determine

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \qquad (\alpha_i \in k, i \in \mathbb{N}_3)$$

from (4.12.1), i.e., from the relation

$$(S - S^t)x = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} x = x \times v\,. \qquad (x \in k^3)$$

In particular,

$$\begin{aligned}
-\alpha e_2 - \beta e_3 &= (S - S^t)e_1 = e_1 \times v \\
&= \alpha_2 e_3 - \alpha_3 e_2 \,, \\
\alpha e_1 - \gamma e_3 &= (S - S^t)e_2 = e_2 \times v \\
&= -\alpha_1 e_3 + \alpha_3 e_1 \,,
\end{aligned}$$

and we conclude $\alpha_1 = \gamma, \alpha_2 = -\beta, \alpha_3 = \alpha$, hence

$$v = v(k^3, \langle S\rangle, \Delta^3) = \gamma e_1 - \beta e_2 + \alpha e_3 \,. \tag{4.12.3}$$

This implies

$$\begin{aligned}
f(v,v) = v^t S v &= v^t \begin{pmatrix} \xi_1 & \alpha & \beta \\ 0 & \xi_2 & \gamma \\ 0 & 0 & \xi_3 \end{pmatrix} \begin{pmatrix} \gamma \\ -\beta \\ \alpha \end{pmatrix} \\
&= (\gamma, -\beta, \alpha) \begin{pmatrix} \xi_1 \gamma \\ -\xi_2 \beta + \alpha\gamma \\ \xi_3 \alpha \end{pmatrix}
\end{aligned}$$

and we conclude

$$f(v,v) = \xi_1\gamma^2 + \xi_2\beta^2 + \xi_3\alpha^2 - \alpha\beta\gamma \,. \tag{4.12.4}$$

c) In b) we specialize the base ring k to

$$k = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{F}_2 \oplus \mathbb{F}_3$$

and the matrix S to

$$S = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(k) \,.$$

Then (4.12.4) implies $f(v,v) = 9 \notin k^\times$, while $4 - f(v,v) = -5 \in k^\times$, and we have realized the special case alluded to in 4.11. Notice that

$$v = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \qquad\qquad \text{(by (4.12.3))}$$

vanishes over $\mathbb{F}_3$, so the polar decomposition of $C = C(k^3, \langle S\rangle, \Delta^3)$ induced by S becomes standard over $\mathbb{F}_3$, while it cannot be standard over $\mathbb{F}_2$, the latter field having characteristic two. Incidentally, since all quaternion algebras over a finite field are split, and since $\operatorname{Spec} k$ is a set of two points carrying the discrete (Zariski) topology, it is clear that $C \cong \mathrm{Mat}_2(k)$ splits over k as well.

4.13. Scaling. Theorem 4.9 above has exhibited two conditions for $C(M, f, \Delta)$ as defined in 4.3 to be a quaternion algebra. One of these, namely,

$$\det\nolimits_\Delta f = 1 \,, \tag{4.13.1}$$

can be fulfilled quite easily by scaling. Indeed, given $\alpha, \beta \in k^\times$, Fact 2.5 gives

$$\det_{\alpha\Delta}(\beta f) = \alpha^{-2}\beta^3(\det_\Delta f)\,, \tag{4.13.2}$$

and by setting $\alpha = \beta = \det_\Delta f$, we may always assume that (4.13.1) holds. This is the reason why the construction around [6, Proposition 2.7] does not require any determinant conditions.

On the other hand, once the normalization (4.13.1) has been carried out, controlling the second condition of Theorem 4.9, namely,

$$4 - f(v, v) \in k^\times\,, \qquad (v = v(M, f, \Delta)) \tag{4.13.3}$$

becomes a much more delicate task. At least, scaling alone won't do. For one thing, (4.13.1), (4.13.2) imply, for all $\alpha, \beta \in k^\times$, that the following statements are equivalent.

(i) $\det_{\alpha\Delta}(\beta f) = 1$.

(ii) $\alpha^{-2}\beta^3 = 1$.

(iii) Some $\gamma \in k^\times$ has $\alpha = \gamma^3, \beta = \gamma^2$ (necessarily, $\gamma = \alpha\beta^{-1}$).

For another, according to (4.3.1), replacing Δ by $\Delta' = \alpha\Delta$, f by $f' = \beta f$ amounts to replacing $v = v(M, f, \Delta)$ by $v' = v(M, f', \Delta') = \alpha^{-1}\beta v$, whence (ii) above yields $f'(v', v') = f(v, v)$. Therefore, in the presence of (4.13.1), if (4.13.3) holds for (M, f, Δ), so it does for (M, f', Δ') and conversely.

4.14. Concluding remarks. Theorem 4.9 may be the final result of the paper, but most likely is not the final word on the subject. From a purely technical point of view, the most important hypothesis that, beginning with 4.3, keeps our approach going, is the non-singularity of f. However, this hypothesis, which by Proposition 3.8 ties up with distinguished rather than arbitrary complements of 1 in quaternion algebras, seems to be rather unnatural since what really counts is, instead, the non-singularity of N_C on all of $C = C(M, f, \Delta)$. Unfortunately, if f is singular, the vector product associated with f and Δ, with all the nice properties needed to carry on, has to be defined by different means. Whether this is really possible, remains an open question up to now.

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