Composition algebras over local fields revisited

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After we have had a great time last night, and also the night before, I suppose it's pretty difficult for each one of us getting back to normal mode. But I can assure you I will do everything I can to make the necessary transition as smooth as possible. Before doing so, however, I strongly believe that Michel as our main host, Erhard as the principal organizer of this event, and Henry Wong as our secondary host, deserve a standing ovation.

If you look at Michel's mathematical work up to now, you will notice that a substantial part if not all of it is dominated by the following two features. The first feature is an all-embracing love for Albert algebras. The second feature, intertwined with the first one in various ways, is an equally all-embracing urge to understand the mysteries of characteristic two.

Unfortunately, this lovely workshop comes just a few months too early to allow me saying anything new about Albert algebras. But it comes at exactly the right time to allow me giving a new twist to the mysteries of characteristic two. I will do so by turning a few stones that I had left unturned more than thirty years ago, in a paper of 1974, entitled *Composition algebras over a field with a discrete valuation*. In order to describe the situation I was interested in at the time, and, indeed, I will be interested in today, let me briefly remind you of a few basic facts about

1. Local fields.

By a *local field* I mean a pair (k, λ) consisting of a field k and a *discrete valuation* λ on k, so $\lambda : k \to \mathbb{R}_\infty$ from k to the reals enlarged by an additional symbol ∞ is a map satisfying the following four conditions, for all $a, b \in k$.

(i) λ is *definite*, so $\lambda(a) = \infty \iff a = 0$.

(ii) λ is *multiplicative*, so $\lambda(ab) = \lambda(a) + \lambda(b)$.

(iii) λ is *sub-additive*, so $\lambda(a + b) \geq \min\{\lambda(a), \lambda(b)\}$.

(iv) λ is *discrete*, so the finite values of λ, denoted by $\Gamma_k := \lambda(k^\times) \subseteq \mathbb{R}$, dependence on λ here and always being understood, form a non-trivial additive subgroup of the reals that is discrete with respect to the natural topology: there exists a unique positive real number q such that $\Gamma_k = \mathbb{Z}q$ is the infinite cyclic group generated by q.

There is, of course, no harm in assuming $q = 1$, which I will do most of the time. Then λ takes values in the integers and is said to be *normalized.*

While all this does not finish the definition of a local field, it already allows me to make a number of

1.1. Important observations. The set

$$\mathfrak{o} := \{a \in k \mid \lambda(a) \geq 0\}$$

is a subring of k, called its *valuation ring.* Indeed, $\mathfrak{o}$ is well known to be a discrete valuation ring in the sense that it is a principal ideal domain containing a unique non-zero prime ideal, denoted by

$$\mathfrak{p} := \{a \in k \mid \lambda(a) > 0\}$$

and called the *valuation ideal* of k, even though, of course, it is only an ideal in $\mathfrak{o}$. In particular, $\mathfrak{o}$ is a local ring with maximal ideal $\mathfrak{p}$, and the quotient

$$\kappa := \mathfrak{o}/\mathfrak{p}$$

is called the *residue field* of k. The natural map from $\mathfrak{o}$ to κ will always be indicated by $a \mapsto \bar{a}$.

This being so, it is straightforward to check that there exists a unique topology on k making it a topological field and having the totality of subsets

$$\mathfrak{p}^d := \{a \in k \mid \lambda(a) \geq d\} \qquad (d \in \mathbb{Z}),$$

λ assumed to be normalized, as a fundamental system of neighborhoods of zero. We say that k or, more precisely, the pair (k, λ), is a *local field* if k is complete with respect to the topology induced by λ in this way. The property of completeness has a number of very nice consequences. For example, local fields satisfy the various equivalent versions of Hensel's Lemma.

Let me now consider

1.2. Basic examples of local fields. There are two cases, the equicharacteristic case and the non-equicharacteristic case. In the

Equicharacteristic case, the residue field κ has the same characteristic as the original local field k. All local fields of this type arise in the following manner.

Let κ be any field and $\mathbf{t}$ a variable. We denote by $k := \kappa((\mathbf{t}))$ the field of formal Laurent series in $\mathbf{t}$ with coefficients in κ and define $\lambda\colon\ k \to \mathbb{Z}_\infty$ by

$$\lambda(u) := \inf\bigl(\{n \in \mathbb{Z} \mid a_n \neq 0\}\bigr) \qquad \Bigl(u = \sum_{n=-\infty}^{\infty} a_n \mathbf{t}^n \in k\Bigr),$$

where the greatest lower bound is always attained unless $u = 0$. One checks easily that λ is a (normalized) discrete valuation making k a local field with valuation ring $\mathfrak{o} = \kappa[[\mathbf{t}]]$, the formal power series ring in $\mathbf{t}$ with coefficients in κ, valuation ideal $\mathfrak{p} = \mathbf{t}\kappa[[\mathbf{t}]]$ and residue field κ. By contrast, the

Non-equicharacteristic case, where k and κ have distinct characteristics, is more delicate; it can arise only if k has characteristic zero and κ is a field of positive characteristic. The key notion in this context is

1.3. The absolute ramification index. Let k be a local field with normalized discrete valuation $\lambda\colon\ k \to \mathbb{Z}_\infty$ and suppose κ has characteristic $p > 0$. If k has characteristic zero, we are in the non-equicharacteristic case, and the prime $p \in \mathbb{Z} \subseteq \mathfrak{o}$ is killed by the natural map from $\mathfrak{o}$ to κ since κ has characteristic p. Thus $p \in \mathfrak{p}$, forcing

$$e_k := \lambda(p) > 0$$

to be a positive integer, called the *absolute ramification index* of k. On the other hand, if k has characteristic p, we are in the equicharacteristic case and formally put

$$e_k := \infty.$$

One of the most important results for non-equicharacteristic local fields is the following existence and uniqueness theorem.

1.4. Theorem. (Witt−Teichmüller 1937) *Given a field κ of positive characteristic, there exists a local field k that up to isomorphism is uniquely determined by the following three conditions.*

(a) *k has characteristic zero.*

(b) *The residue field of k is κ.*

(c) *k is* absolutely unramified, *that is, $e_k = 1$.*

1.5. Example. Let p be a prime. Then $\mathbb{Q}_p$, the field of p-adic numbers, is the unique absolutely unramified local field of characteristic zero having residue field $\mathbb{F}_p$.

I can now turn to the main topic of this lecture, namely,

2. Composition algebras over local fields.

Let k be a local field, with normalized discrete valuation $\lambda\colon\ k \to \mathbb{Z}_\infty$, and C a composition algebra over k, with norm n_C, trace t_C and conjugation $x \mapsto x^*$. By the general theory of composition algebras over arbitrary fields, composition algebras with zero divisors over k are uniquely determined by their dimension, and their structure is explicitly known. Therefore we may and always will assume that C is a division algebra. Our aim is understand C by means of three invariants that are *local* in the sense that they are intimately tied up with the structure of k as a local field. While two of these invariants belong to the standard set of tools provided by valuation theory and, in fact, have already been put to good use in my paper of 1974, the third one, though actually pretty obvious, does not seem to have been considered before. All three invariants are based on the following extension theorem.

2.1. Theorem. (a) *The map*

$$\lambda_C\colon\ C \longrightarrow \mathbb{R}_\infty, \quad x \longmapsto \frac{1}{2}\lambda\big(n_C(x)\big)$$

is the unique discrete valuation of C extending λ. It thus gives rise to

- $\mathfrak{o}_C := \{x \in C \mid \lambda_C(x) \geq 0\} \subseteq C$, *an $\mathfrak{o}$-subalgebra, the* valuation ring *of C,*
- $\mathfrak{p}_C := \{x \in C \mid \lambda_C(x) > 0\} \subseteq \mathfrak{o}_C$, *a two-sided ideal, the* valuation ideal *of C,*
- $\bar{C} := \mathfrak{o}_C/\mathfrak{p}_C$, *a division algebra over κ, the* residue algebra *of C,*
- $x \longmapsto \bar{x}$, *the natural map from $\mathfrak{o}_C$ to $\bar{C}$.*

(b) *The map*

$$n_{\bar{C}} : \bar{C} \longrightarrow \kappa, \quad \bar{x} \longmapsto n_{\bar{C}}(\bar{x}) := \overline{n_C(x)},$$

is a well defined anisotropic quadratic form over κ permitting composition in the sense that $n_{\bar{C}}(\bar{x}\,\bar{y}) = n_{\bar{C}}(\bar{x})n_{\bar{C}}(\bar{y})$ for all $x, y \in \mathfrak{o}_C$. □

2.2. Corollary. *Either $\bar{C}$ is a composition division algebra over κ in the sense that the polar of its norm is a non-degenerate symmetric bilinear form, or $\bar{C}/\kappa$ is a purely inseparable field extension of characteristic* 2 *and exponent at most* 1.

□

Even if k has characteristic zero and C is an honest-to-goodness octonion division algebra over k, we will see in due course that the second alternative in the preceding corollary may very well occur. This gives you a first indication as to what kind of mysteries of characteristic two I have in mind. But before pursuing this topic any further, let us first turn to

2.3. The local invariants. (a) $\mathbb{Z} = \Gamma_k \subseteq \Gamma_C := \lambda_C(C^\times) \subseteq \frac{1}{2}\mathbb{Z}$ is a chain of subgroups which shows

$$e_C := [\Gamma_C : \Gamma_k] \in \{1, 2\}.$$

We call e_C the *ramification index* of C.

(b) $f_C := \dim_\kappa(\bar{C})$ is called the *residue degree* of C. We have the fundamental relation

$$e_C f_C = \dim_k(C).$$

(c) We say C is *unramified* (resp. *ramified*) if $e_C = 1$ (resp. $e_C = 2$) and $\bar{C}$ is a non-singular composition algebra over κ. Since a detailed description of ramified and unramified composition division algebras may already be found in my paper of 1974, they will not concern us here. Instead, we will focus attention on the remaining case of composition division algebras C over k that are *residually inseparable* in the sense that $\bar{C}/\kappa$ is a purely inseparable field extension of characteristic 2 and exponent at most 1. They will be investigated here with the aid of yet another local invariant.

(d) $t_C(\mathfrak{o}_C) \subseteq \mathfrak{o}$ is an ideal, so there exists a unique element $r_C \in \mathbb{N}^0_\infty := \mathbb{N} \cup \{0, \infty\}$ such that

$$t_C(\mathfrak{o}_C) = \mathfrak{p}^{r_C}$$

(where we agree to the convention $\mathfrak{p}^\infty := \{0\}$.) We call r_C the *trace exponent* of C. It is easy to see that

$$0 \le r_C \le e_k \quad \text{and} \quad r_k = e_k,$$

where e_k stands for the absolute ramification index of k. Moreover,

$$r_C = 0 \Longleftrightarrow C \text{ is ramified or unramified.}$$

Our main concern in this lecture will be to understand the behavior of the local invariants $e_C, \bar{C}, r_C$ under the influence of the Cayley-Dickson construction. For this purpose, we require a technical tool called

2.4. The norm exponent. In his book of 1963 on quadratic forms, O'Meara introduces the concept of *quadratic defect* in order to describe the square classes of local fields having characteristic not 2 and a finite residue field. It turns out that his approach can be extended to composition division algebras over arbitrary local fields as follows.

Till the end of this lecture, we fix a prime element $\pi \in \mathfrak{o}$. Consider a non-singular composition division algebra B over k, a unit $a \in \mathfrak{o}^\times$ and a non-negative integer $d \in \mathbb{N}^0$. Then it is straightforward to check that the following conditions are equivalent:

(i) *a is a norm of B modulo $\mathfrak{p}^d$*, so there exists a $v \in \mathfrak{o}_B^\times$ such that $a - n_B(v) \in \mathfrak{p}^d$.

(ii) There exist $b \in \mathfrak{o}$, $v \in \mathfrak{o}_B^\times$ satisfying

$$a = (1 - \pi^d b) n_B(v).$$

We put

$$N_B(a) := \{d \in \mathbb{N}^0 \mid d \text{ satisfies (i)/(ii)}\} \subseteq \mathbb{N}^0$$

and note

$$0 \in N_B(a), \quad \forall d, d' \in \mathbb{N}^0 \Big(d \in N_B(a),\, d' \leq d \Longrightarrow d' \in N_B(a) \Big).$$

Then

$$\mathrm{nex}_B(a) := \sup\big(N_B(a)\big) \in \mathbb{N}^0_\infty$$

is called the *norm exponent* of a relative to B. For example, if $e_B = 1$, it is easily checked that

$$\mathrm{nex}_B(a) = 0 \Longleftrightarrow \bar{a} \notin n_{\bar{B}}(\bar{B}^\times).$$

Moreover, in the special case $B = k$ ($\mathrm{char}(k) \neq 2$), $\mathfrak{p}^{\mathrm{nex}_B(a)}$ is the quadratic defect of a in the sense of O'Meara. The key result about the norm exponent is the following generalization of O'Meara's Local Square Theorem.

2.5. Local Norm Theorem. *For $a \in \mathfrak{o}^\times$, the following conditions are equivalent.*

(i) $a \in n_B(B^\times)$.

(ii) $a \in n_B(\mathfrak{o}_B^\times)$.

(iii) $\mathrm{nex}_B(a) \geq 2r_B + 1$.

□

The only non-trivial part of the proof is the implication (iii) ⇒ (i).

We are now in a position to deal with

2.6. The local invariants under the Cayley-Dickson construction. We fix a composition division algebra C over k and a non-singular composition subalgebra $B \subseteq C$ such that

$$\dim_k(B) = \frac{1}{2}\dim_k(C) \quad \text{and} \quad e_B = 1.$$

(Such a B always exists unless C is two-dimensional and k has characteristic 2.) Then $C = \mathrm{Cay}(B, \mu)$ for some scalar $\mu \in k^\times$, and since we are allowed to multiply μ by a non-zero norm of B, in particular, by a non-zero square in k, without changing the isomorphism class of C, we may assume

$$\mu \in \mathfrak{o} \text{ is a prime or a unit.}$$

We now distinguish the following cases.

Case 1. $\bar{B}$ is a non-singular composition algebra over κ. Then

- $r_B = r_C = 0$.
- $e_C = 1 \Longleftrightarrow C$ is unramified $\Longleftrightarrow \mu \in \mathfrak{o}^\times$ is a unit, in which case $\bar{C} = \mathrm{Cay}(\bar{B}, \bar{\mu})$.
- $e_C = 2 \Longleftrightarrow C$ is ramified $\Longleftrightarrow \mu \in \mathfrak{o}$ is a prime, in which case $\bar{C} = \bar{B}$.

The last two statements are already in my paper of 1974.

Case 2. $\bar{B}/\kappa$ is a purely inseparable field extension of characteristic 2 and exponent at most 1. Then $r_B > 0$.

Case 2.1. (easy) $\mu \in \mathfrak{o}$ is a prime. Then

$$e_C = 2, \quad \bar{C} = \bar{B}, \quad r_C = r_B.$$

Case 2.2. $\mu \in \mathfrak{o}^\times$ is a unit. Since C is a division algebra, μ is not a norm of B, hence by the Local Norm Theorem has norm exponent $d = \mathrm{nex}_B(\mu) \le 2r_B$ relative to B. Thus $\mu = (1 - \pi^d b)n_B(v)$ for some $b \in \mathfrak{o}$, $v \in \mathfrak{o}_B^\times$, and we may assume $v = 1_B$. Summing up, we are reduced to the case

$$\mu = 1 - \pi^d b$$

for some $b \in \mathfrak{o}$ and some $d \in \mathbb{N}^0$ satisfying $0 \le d \le 2r_B$. We do *not* assume that $d = \mathrm{nex}_B(\mu)$ is the norm exponent of μ relative to B but, on the contrary, want to obtain conditions that are necessary and sufficient for this to happen.

Case 2.2.1. d is *odd*. Then

$$d = \mathrm{nex}_B(\mu) \Longleftrightarrow b \in \mathfrak{o}^\times.$$

In this case,

$$e_C = 2, \quad \bar{C} = \bar{B}, \quad r_C = r_B - \frac{d-1}{2}.$$

Moreover, if also $b' \in \mathfrak{o}$, $\mu' := 1 - \pi^d b'$, then

$$\mathrm{Cay}(B, \mu) \cong \mathrm{Cay}(B, \mu') \Longrightarrow \bar{b} = \overline{b'}.$$

But the converse is *false*.

Case 2.2.2. d is *even* and $d < 2r_B$. Then

$$d = \mathrm{nex}_B(\mu) \Longleftrightarrow \bar{b} \in \kappa \setminus \overline{B}^2.$$

In this case,

$$e_C = 1, \quad \bar{C} = \bar{B}(\sqrt{\bar{b}}), \quad r_C = r_B - \frac{d}{2}.$$

Moreover, if also $b' \in \mathfrak{o}$, $\mu' := 1 - \pi^d b'$, then

$$\mathrm{Cay}(B, \mu) \cong \mathrm{Cay}(B, \mu') \Longrightarrow \bar{b} \equiv \overline{b'} \bmod \overline{B}^2.$$

Again, the converse is *false.*

Case 2.2.3. $d = 2r_B$. This is the most interesting case of them all. Let me simplify matters a bit by restricting myself to dimension 8, so C is an octonion division algebra containing B as a quaternion subalgebra. In fact, C turns out to be unramified over k, forcing $\bar{C}$ to be an octonion division algebra over κ containing $\bar{B}$ as maximal inseparable subfield of degree 4, and all of a sudden we are confronted with a set-up that hasn't got anything to do with local fields. The appropriate way to understand this set-up consists in looking at what I call

2.7. The non-orthogonal Cayley-Dickson construction. Let F be any field of characteristic 2, A be any octonion algebra (possibly split) over F, and $L \subseteq A$ a (maximal) purely inseparable subfield of degree 4. The question is how to describe the structure of A in terms of L and, possibly, a bunch of additional data. What is needed to achieve this are

- an arbitrary scalar $\gamma \in F$,

- a linear form $\delta\colon L \to F$ which is *unital* in the sense that $\delta(1_L) = 1$.

Once these data have been selected, the non-orthogonal Cayley-Dickson construction leads to an octonion algebra $A = \mathrm{Cay}(L; \gamma, \delta)$, and every octonion algebra over F containing L as a unital subalgebra arises in this manner. Moreover, A is a division algebra if and only if γ does not belong to the range of the *Artin-Schreier* map

$$\wp_{L,\delta}\colon\ L \longrightarrow F, \quad u \longmapsto \wp_{L,\delta}(u) := u^2 + \delta(u).$$

Returning to our local field k, the non-orthogonal Cayley-Dickson construction enters

Case 2.2.3. (cont'd) in the following way. Recall from the definition of the trace exponent that $t_B(\mathfrak{o}_B) = \mathfrak{p}^{r_B}$, so we can find an element $w_0 \in \mathfrak{o}_B$ satisfying $t_B(w_0) = \pi^{r_B}$, and it is easy to see that this implies $w_0 \in \mathfrak{o}_B^\times$. We put $b_0 := n_B(w_0)b$ and consider the linear form

$$\delta\colon\ B \longrightarrow k, \quad u \longmapsto \pi^{-r_B} n_B(w_0, u),$$

which is easily seen to induce canonically a unital linear form $\bar{\delta}\colon\ \bar{B} \to \kappa$. Then

$$\mathrm{nex}_B(\mu) = 2r_B \Longleftrightarrow \overline{b_0} \notin \mathrm{Im}(\wp_{\bar{B},\bar{\delta}});$$

in this case

$$e_C = 1, \quad \bar{C} = \mathrm{Cay}(\bar{B}; \overline{b_0}, \bar{\delta}), \quad r_C = 0.$$

What are the applications of the preceding results?

For one, the local invariants may be arbitrarily pre-assigned in advance as long as they satisfy a few rather obvious constraints. More precisely, we have the following

2.8. Theorem. *Let $e, r \in \mathbb{Z}$, $n \in \mathbb{N}^0$ and A a κ-algebra of dimension 2^n. Then the following conditions are equivalent.*

(i) *There exists a composition division algebra C over k having local invariants $e_C = e$, $\bar{C} \cong A$, $r_C = r$.*

(ii) (a) $e \in \{1, 2\}$, $0 \le n \le 4 - e$, $0 \le r \le e_k$.
   (b) *A is either a non-singular composition division algebra over κ or a purely inseparable field extension of characteristic 2 and exponent at most 1.*
   (c) *A is a non-singular composition algebra if and only if $r = 0$.*

□

For another application, the local invariants are lightyears away from classifying composition algebras over local fields. Indeed, using our previous results, it is easy to construct examples for the following situation: k is a local field, L/k is a purely inseparable field extension of degree 8 and exponent 1, r is an integer such that $0 < r < e_k$, and there are an infinite number of mutually non-isomorphic octonion division algebras C over k having local invariants $e_C = 1$, $\bar{C} \cong L$, $r_C = r$. By contrast, *unramified* octonion division algebras over k (which correspond to the case $r = 0$) are known to be uniquely determined by their residue algebras.

Let me close this lecture by stating an

2.9. Open question. Composition algebras over any field, in particular those over our local field k (for simplicity assumed to be of characteristic not 2), are classified by their dimension (2^n) and a single cohomological invariant belonging to $H^n(k, \mathbb{Z}/2\mathbb{Z})$ in the sense of Galois cohomology. It would be interesting to know how this cohomological invariant of a composition division algebra over k relates to its local invariants.