ADER methods and generalized Riemann problems for hyperbolic conservation laws

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Hyperbolic conservation laws arise in many relevant applications, ranging gas dynamics to the modelling of traffic flow. We are concerned with the Cauchy-problem for systems of conservation laws in one spatial dimension,

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) &= u_0(x), & x \in \mathbb{R}, \end{aligned}$$

where the solution $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$ is a vector of conserved quantities (such a mass, momentum and energy in hydrodynamics) and $f : \mathbb{R}^m \to \mathbb{R}^m$ is the (physical) flux. A predominant feature of nonlinear hyperbolic conservation laws is the development of shock waves, i.e. the solution u may become discontinuous in finite time, even for arbitrary smooth flux f and initial data u_0.

A recent approach for the numerical solution of hyperbolic conservation laws is the *ADER method*. The ADER method is a high order finite volume method and can be interpreted as a generalization of the classic Godunov method. The scheme relies on two ingredients: At each time-step,

- build a piecewise smooth approximation to the solution u. Usually this is done by a WENO reconstruction from cell-averages;
- solve a Cauchy problem with piecewise polynomial data that can be discontinuous at the cell-interfaces. This is called the generalized Riemann problem (GRP).

We describe a well-known method for the approximate solution of the GRP that is based on the solution of a series of classical Riemann problems. This GRP-solver has been successfully used in numerical applications. However, it seems that few analysis of its theoretical properties has been done so far. We use an asymptotic expansion of the solution of the GRP to show that (at least for scalar problems) the solution methodology can be derived rigorously.