Fast computation of optimal quadrature nodes on the sphere

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In this talk we present an efficient optimization approach for distributing M points in some optimal sense on the sphere $\mathbb{S}^2$. Therefor we consider the polynomial reproducing kernel Hilbert space $\Pi_{K_N}(\mathbb{S}^2) = \operatorname{span}\{Y_n^k : n = 0, \ldots, N;\ k = -n, \ldots, n\}$ with reproducing kernel

$$K_N(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=0}^{N} \lambda_n^{-1} \sum_{k=-n}^{n} Y_n^k(\boldsymbol{x}) \overline{Y_n^k(\boldsymbol{y})}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^2, \quad \lambda_n > 0,$$

where Y_n^k are the spherical harmonics of degree n and order k. In this space we are interested in approximating the integrals

$$I(f) = \int_{\mathbb{S}^2} f(\boldsymbol{x}) \mathsf{d}\mu_{\mathbb{S}^2}(\boldsymbol{x})$$

for $f \in \Pi_{K_N}(\mathbb{S}^2)$ by a quadrature rule

$$Q(f, \mathcal{X}) = \frac{4\pi}{M} \sum_{i=1}^{M} f(\boldsymbol{x}_i)$$

for an appropriately chosen point set $\mathcal{X} := \{\boldsymbol{x}_i \ : \ i = 1, \ldots, M\} \subset \mathbb{S}^2$. In order to do this in an optimal way, we aim to minimize the worst case quadrature error

$$\mathsf{err}^2_{K_N}(\mathcal{X}) := \sup_{\substack{f \in \Pi_{K_N}(\mathbb{S}^2) \\ \|f\|_{\Pi_{K_N}(\mathbb{S}^2)} \leq 1}} |I(f) - Q(f, \mathcal{X})|^2 = \sum_{n=1}^{N} \sum_{k=-n}^{n} \lambda_n^{-1} \left| \frac{4\pi}{M} \sum_{i=1}^{M} Y_n^k(\boldsymbol{x}_i) \right|^2$$

over all point sets $\mathcal{X}$ with M points. The proposed optimization scheme is based on a nonlinear conjugate gradient method for Riemannian manifolds [2]. We show that every iteration step of this algorithm is performed in $\mathcal{O}(N^2 \log^2 N + M)$ arithmetic operations by means of the nonequispaced fast spherical Fourier transform [1]. At the end we give some numerical examples which demonstrate for large M and N the good performance of the algorithm.

[1] J. Keiner, S. Kunis, and D. Potts. Using NFFT3 - a software library for various nonequispaced fast Fourier transforms. *ACM Trans. Math. Software*, 36:Article 19, 1 – 30, 2009.

[2] S. T. Smith. Optimization techniques on Riemannian manifolds. In *Hamiltonian and gradient flows, algorithms and control*, volume 3 of *Fields Inst. Commun.*, pages 113 – 136. Amer. Math. Soc., Providence, RI, 1994.

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