Average-case modelling of the roundoff error for several classes of fast trigonometric transforms

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Discrete sine and cosine transforms (DSTs & DCTs), as classified in [5], belong to a special kind of matrix-vector multiplication. They are closely connected to the discrete Fourier transform (DFT) and have a wide range of application [8], not only since they can be easily inverted due to the orthogonality of the underlying transform matrices. Similarly to the well-known [1, 9] fast Fourier transform (FFT), fast and numerically stable algorithms can be built for DCTs and DSTs [4] by using the 'divide-and-conquer idea', especially in case of transform length $n = 2^t$ ($t \in \mathbb{N}$). Since in digital signal processors (DSPs), both floating-point and fixed-point arithmetic are commonly used, the roundoff error produced by the considered algorithms, needs to be analyzed in both arithmetics.

Concerning the behavior of the roundoff error, Higham [1] states that '[a fast algorithm] is best understood (at least by a numerical analyst!) by interpreting it as the application of a clever factorization of the [...] transform matrix'. Within the representation

$$A \;=\; \prod_{m=1}^{\nu} A^{(m)} \;:=\; A^{(\nu)} \dots A^{(2)} A^{(1)} \tag{1}$$

of the corresponding orthogonal transform matrix A, which each of the DCT- and DST-algorithms introduced in [4, 6] is based on, every factor $A^{(m)}$ is both orthogonal and *sparsely populated*, i.e., it possesses at most two non-zero entries in every row and in every column. By using only permutations and row- or column-wise sign inverting if necessary, every $A^{(m)}$ in (1) can be additionally converted into a block-diagonal matrix, where every block

$$Q_2(\varphi) \;:=\; \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \qquad \left(\varphi \in \left[0, \tfrac{\pi}{4}\right]\right) \tag{2}$$

represents a plane rotation. However, permutations and sign inversion do not produce any additional roundoff error. Therefore, it suffices to study structure (2).

Next to worst-case bounds [6, 7, 4, 2], we are interested in average-case predictions for the roundoff error. Since multiplication of two complex numbers can be interpreted as a scaled plane rotation, the roundoff-error model introduced in [10] has turned out to be compatible to structure (2) and – after slight modifications [3] which allow application in series – it has proved to be applicable to any algorithm which is based on a factorization (1) whose factors can be converted in the way described above into block-diagonal matrices with blocks (2).

Literatur

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