Albert algebras arising from central simple quartic Jordan algebras are reduced.

Holger P. Petersson
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
D-58084 Hagen, Germany
Email: holger.petersson@FernUni-Hagen.de

1. Introduction. A method of constructing cubic Jordan algebras out of quartic ones, akin to the approach of Okubo [4, 5] and Faulkner [2] for finding symmetric compositions inside associative algebras of degree 3, has been devised by Allison-Faulkner [1]. Their key results (as far as the scope of the present note is concerned) read as follows. Throughout we let $k$ be a field of characteristic not 2 or 3.

2. Theorem. (Allison-Faulkner [1, Theorem 5.4]) Let $J$ be a separable Jordan algebra of degree 4 over $k$, write $t$ for the (generic) trace of $J$ and $J_0 := \{x \in J \mid t(x) = 0\}$ for the linear hyperplane of trace zero elements in $J$. Suppose we are given an element $e \in J_0$ satisfying $t(e^3) \neq 0$. Then there is a unique way of making $J_0$ into a separable Jordan algebra of degree 3 over $k$ having unit element $e$ and generic norm given by

$$N_0(x) = \frac{t(x^3)}{t(e^3)} \quad (x \in J_0).$$

This Jordan algebra will be denoted by $\text{Zer}(J, e)$. □

3. Corollary. (Allison-Faulkner [1, Corollary 5.5]) Let $(A, \tau)$ be a central simple associative algebra of degree 8 with symplectic involution over $k$ and $e \in A$ a $\tau$-symmetric element of trace zero satisfying $t(e^3) \neq 0$, where $t$ stands for the generic trace of $A$. Then $\text{Zer}(A, \tau, e) := \text{Zer}(\text{H}(A, \tau), e)$ is an Albert algebra over $k$. □

In a recent preprint, working in a much more general context, S. Pumplün [7] has described it as unclear whether all Albert algebras over $k$ can be obtained in the manner described by Corollary 3. Using standard techniques, it is, however, possible to derive the following result.

4. Theorem. Let $(A, \tau)$ be a central simple associative algebra of degree 8 with symplectic involution over $k$ and $e \in A$ a $\tau$-symmetric element of trace zero satisfying $t(e^3) \neq 0$, where $t$ stands for the generic trace of $A$. Then the Albert algebra $\text{Zer}(A, \tau, e)$ is reduced.

Proof. We put $J := \text{Zer}(A, \tau, e)$ and make use of its cohomological mod-3-invariant belonging to $H^3(k, \mathbb{Z}/3\mathbb{Z})$, see Rost [8], Garibaldi-Merkurjev-Serre [3] or Petersson-Racine [6] for details. Accordingly, if $J$ is a division algebra, so is $J \otimes_k l$, for any finite algebraic field extension $l/k$ of degree prime to 3. Now the division algebra belonging to the Brauer class of $A$ has degree $2^n$, $0 \leq n \leq 3$. Hence $A$ has a separable splitting field whose degree is a power of 2. By what has just been observed, we may therefore assume that $(A, \tau)$ is split, i.e., $A = \text{Mat}_8(k)$ and $\tau$ is the symplectic involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$$

1
in terms of $2 \times 2$-blocks of $4 \times 4$-matrices. It follows that $H(A, \tau)$ consists of the matrices
\[
\begin{pmatrix}
a & b \\
c & a^t
\end{pmatrix}
(a \in \text{Mat}_4(k), b, c \in \text{Skew}_4(k)).
\]
Picking a non-zero nilpotent $a \in \text{Mat}_4(k)$ of index at most 3, we conclude that
\[x := \begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix} \in H(A, \tau)\]
has trace zero and satisfies $x^3 = 0$, hence $N_J(x) = 0$. Thus $J$ is not a division algebra. □

References


