On Voting Systems with Correlated Voting Behavior

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Stochastic Analysis and Related Fields; Beijing, May 2010
The well being of democracies regardless of their type and status depends on a small technical detail: *the voting system.*

Everything else is secondary.

*Jose Ortega y Gasset*
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Voting Systems (Parliaments)

National People’s Congress

German Bundestag

US-House of Representatives
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‘Normally’ a proposal is adopted if the number of votes in favor of the proposal is bigger than the number of votes against it (simple majority).
A parliament is a **voting system**. The members of the parliament are the **voters** of this voting system.

Each voter may vote ‘Yes’ or ‘No’.

Here and in what follows we neglect the possibility of abstentions.

‘Normally’ a proposal is adopted if the number of votes in favor of the proposal is bigger than the number of votes against it (**simple majority**).

In some cases a **qualified majority** is required (e.g. two thirds), for example to make an amendment to the constitution.
UN security council

- **Five permanent members**
  (China, France, Russia, UK, USA)

- **Ten non permanent members**
  (Austria, Bosnia-Herzegovina, Brasil, Gabon, Japan, Lebanon, Mexico, Nigeria, Turkey, Uganda)

- A proposal is approved if **both**
  - All permanent members affirm it and
  - At least **four** non permanent members do.
In 1958 six European States founded the EEC, the predecessor of the EU. The ruling body was the Council of Ministers with ‘weighted’ votes:

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</tr>
<tr>
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<td>4</td>
</tr>
<tr>
<td>Luxembourg</td>
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The votes of a country have to be cast in a block, i.e. they cannot be split. The number of votes of a country is called its voting weight.

To approve a proposal 12 (out of 17) were needed. (quota)
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To approve a proposal 12 (out of 17) were needed. (quota)

The voting weight of a member state depends (at least monotonously) on its population.
The International Monetary Fund (IMF) is an organization (a sub-organization of the UN) with 186 member states. It analyzes the economical development worldwide and, if necessary, supports its member states through loans, usually connected with severe constraints on their economic and fiscal policy.
The number of votes of a country in the Board of Governors is connected with the country’s economic strength (through a rather sophisticated formula). Here are the voting weights for a few countries:

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For many important decisions of the IMF a majority of 85 % is required. The US-share is 16.74 % . . .
A voting system consists of a finite set \( V \) of voters and a subset \( \mathcal{W} \) of the power set \( \mathcal{P}(V) \). A subset \( M \) of \( V \) is called a coalition, the sets \( M \) in \( \mathcal{W} \) are called winning coalitions.

We suppose:
- \( V \in \mathcal{W} \) (grand coalition)
- \( \emptyset \notin \mathcal{W} \) (empty coalition)
- If \( A \subset B \) and \( A \in \mathcal{W} \) then \( B \in \mathcal{W} \).

Weighted Voting Systems

A voting system \((V, \mathcal{W})\) is called weighted, if there a function \( w: V \rightarrow \mathbb{R}^+ \) (weight) and a number \( q \) (quota), such that

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A \in \mathcal{W} \iff \sum_{v \in A} w(v) \geq q \sum_{v \in V} w(v)
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Winning coalitions: All 5 permanent members and 4 (or more) nonpermanent members
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The UN Security Council is a weighted voting system, as we may set:

\[ w(x) := \begin{cases} 
7, & \text{for permanent members } x; \\
1, & \text{for nonpermanent members.} 
\end{cases} \]

and

\[ q := 39 \]
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and

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    q := 39
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Not every voting system can be written as a weighted voting system.
The Banzhaf-Index

If $A \in \mathcal{W}$ and $v \in A$ then $v$ is called decisive for the coalition $A$ if $A \setminus \{v\} \not\in \mathcal{W}$.

$$\mathcal{D}(v) := \{A \in \mathcal{W} \mid v \text{ is decisive for } A\}$$

The Banzhaf-Power $B(v)$ of a voter $v$ is defined as

$$B(v) := \frac{1}{2^{\#V-1}} \#\mathcal{D}(v)$$

The Banzhaf-Index $\beta(v)$ of a voter $v$ is defined as

$$\beta(v) := \frac{B(v)}{\sum_{u \in V} B(u)}$$
### Example: Council of Ministers of the EEC

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<tr>
<th>Country</th>
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<tbody>
<tr>
<td>Belgium</td>
<td>11.8%</td>
<td>14.3%</td>
</tr>
<tr>
<td>France</td>
<td>23.5%</td>
<td>23.8%</td>
</tr>
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</tr>
<tr>
<td>Luxembourg</td>
<td>5.9%</td>
<td>0.0%</td>
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Voting Systems  
Beijing, May 2010
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**Quota**: 12
The Shapley-Shubik-Index

For the Shapley-Shubik-Index we consider permutations $\pi \in S_V$ (rather than subsets of $V$).
We say that a voter $v$ is decisive for a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in S_V$, (We set $N := \#V$), if $v = \pi_\nu$ and

$$\{\pi_j \mid j \leq \nu\} \in \mathcal{W} \quad \text{but} \quad \{\pi_j \mid j < \nu\} \notin \mathcal{W}$$

We define the Shapley-Shubik-Index $\sigma(v)$ of a voter $v$ by

$$\sigma(v) := \frac{1}{\#(S_V)} \#\{\pi \in S_V \mid v \text{ is decisive for } \pi\}$$
Power Indices measure ‘how frequently’ it happens that a given voter is decisive.
Power-Indices

Power Indices measure ‘how frequently’ it happens that a given voter is decisive.

We will, in deed, give it a probabilistic interpretation later.
Typical Examples

- Each member state of the EU (or the IMF) is represented by one delegate in the Council of Ministers (or the board of Governors).
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- We assume that the delegates follow the majority vote in their respective country.
Multi-Layer Voting Systems

Typical Examples

- Each member state of the EU (or the IMF) is represented by one delegate in the Council of Ministers (or the board of Governors).
- The delegate of a country has a given voting weight depending on the population (or the economic strength) of his country. This vote cannot be split.
- We assume that the delegates follow the majority vote in their respective country.
- This constitutes a two-layer (or two-step) voting system for the citizens in these countries.
The set of voters consists of $M$ disjoint sets $S_\nu, \nu = 1, \ldots, M$ (states). The set $S_\nu$ contains $N_\nu$ elements.

The vote cast by the voter number $i$ in state $S_\nu$ is denoted by $X_{\nu i}$.

\[ X_{\nu i} = \begin{cases} 
+1, & \text{if he/she votes ‘YES’;} \\
-1, & \text{if she/he votes ‘NO’}.
\end{cases} \]

The popular vote $X_\nu$ in state $S_\nu$ is given by:

\[ X_\nu = \sum_{i=1}^{N_\nu} X_{\nu i} \]

With $\chi(x) = \begin{cases} 
+1, & \text{if } x > 0; \\
-1, & \text{if } x \leq 0.
\end{cases}$ the voting behavior of the delegate of $S_\nu$ is given by

\[ Y_\nu = \chi(X_\nu) \]
If we give the state $S_\nu$ a weight $w_\nu$ then the voting outcome in the Council is

$$\sum_{\nu=1}^{M} w_\nu Y_\nu = \sum_{\nu=1}^{M} w_\nu \chi(\sum_{i=1}^{N_\nu} X_{\nu i})$$
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We consider a two-layer voting system as ‘fair’ if the voting result in the council is as close as possible to the popular vote

$$\sum_{\nu=1}^{M} \sum_{i=1}^{N_\nu} X_{\nu i} \ , \ i.e. \ we \ want \ to \ minimize \ (\text{with \ respect \ to \ the} \ w_\nu):$$

$$\Delta := \left| \sum_{\nu=1}^{M} w_\nu \ \chi\left(\sum_{i=1}^{N_\nu} X_{\nu i}\right) - \sum_{\nu=1}^{M} \sum_{i=1}^{N_\nu} X_{\nu i} \right|$$
If we give the state $S_{\nu}$ a weight $w_{\nu}$ then the voting outcome in the Council is

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The quantity $\Delta$ is called the democracy deficit of the voting system.
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The quantity $\Delta$ is called the democracy deficit of the voting system.
Note, that it is impossible to make $\Delta$ identically equal to zero!
It seems therefore reasonable to minimize

$$\mathbb{E}(\Delta^2) = \mathbb{E}\left( \left( \sum_{\nu=1}^{M} w_{\nu} \chi\left( \sum_{i=1}^{N_{\nu}} X_{\nu i} \right) - \sum_{\nu=1}^{M} \sum_{i=1}^{N_{\nu}} X_{\nu i} \right)^2 \right)$$

which is some kind of mean square error.
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But . . .

. . . what is $\mathbb{E}$ supposed to mean?
The voting system is fed with proposals in a completely random way.
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The voters react to the (random) proposal in a deterministic rational way.
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The randomness of the proposals induce a probability measure $\mathbb{P}$ on the space $\{-1, +1\}^V$. A proposal $\omega$ and its counter-proposal $\neg \omega$ should have the same probability. The rationality of the voting implies that $X_v(\neg \omega) = -X_v(\omega)$. Consequently, the probability $\mathbb{P}$ is invariant under inversion

$$ (X_1, \ldots, X_N) \mapsto (-X_1, \ldots, -X_N) $$

In particular:

$$ \mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2} $$
Independent Voters

If the voting behavior $X_i$ of the voter $i$ is independent from the other voters, then $P$ is just a product measures $P_0$ with $P_0 = \frac{1}{2}(\delta_{-1} + \delta_{+1})$. 

Remark

The Banzhaf $B(v)$ can be expressed through this measure, namely:

$$B(v) = 2 P(D(v))$$
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$$B(\nu) = 2 P(D(\nu))$$
Notation

We set:

$$Q = Q^{(N)} := \bigotimes_{i=1}^{N} \left( \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1} \right)$$

and (to be used later):

$$Q_p = Q_p^{(N)} := \bigotimes_{i=1}^{N} \left( p \delta_1 + (1 - p) \delta_{-1} \right)$$
The CBM

For the **Common Believe Model (CBM)** (or Collective Bias Model (CBM)) we suppose there is a strong common value system, for example there may be a strong church or religious group or a state-wide ideology or the media may be under the control of a small group of people or . . .
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This common believe (or collective bias) is expressed through a random variable $Z$ with values in the interval $[-1, 1]$ and distribution $\mu$. 

\[
P_{\mu}(A) := \int_{[-1, 1]} Q_1(1 + z)(A) d\mu(z)
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If $\mu = \delta_0$ we get back to the independent voters model.
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Remarks

- The measure $\mathbb{P}_\mu$ is spin-flip invariant, if $\mu$ is symmetric (at 0).
- If $\mu$ is the uniform distribution on $[-1, 1]$, then the Shapley-Shubik-Index $\sigma(v)$ can be expressed as:

$$\mathbb{P}_\mu(D(v))$$
We return to the question of fair voting weights.

We recall that we want to chose the weights $w_{\nu}$ in such a way, that the democracy deficit

$$\Delta := \left| \sum_{\nu=1}^{M} w_{\nu} \chi\left( \sum_{i=1}^{N_{\nu}} X_{\nu i} \right) - \sum_{\nu=1}^{M} \sum_{i=1}^{N_{\nu}} X_{\nu i} \right|$$

is as small as possible, in the sense, that we try to minimize

$$\mathbb{E}(\Delta^2) = \mathbb{E}\left( \left( \sum_{\nu=1}^{M} w_{\nu} \chi\left( \sum_{i=1}^{N_{\nu}} X_{\nu i} \right) - \sum_{\nu=1}^{M} \sum_{i=1}^{N_{\nu}} X_{\nu i} \right)^2 \right)$$
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Let us assume, that voters in different states act in an independent way of each others, i.e. $X_{\nu i}$ and $X_{\rho j}$ with $\nu \neq \rho$ are independent of each others.
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But \( X_{\nu i} \) and \( X_{\nu j} \) may be dependent.
The quantity $\mathbb{E}(\Delta^2)$ is minimized by the choice

$$w_{\nu} = \mathbb{E}\left(\left| \sum_{i=1}^{N_{\nu}} X_{\nu i} \right| \right)$$
Result

The quantity $\mathbb{E}(\Delta^2)$ is minimized by the choice

$$w_\nu = \mathbb{E} \left( \left| \sum_{i=1}^{N_\nu} X_{\nu i} \right| \right)$$

Independent Voters

For independent $X_{\nu i}$ we have (for large $N_\nu$):

$$\mathbb{E} \left( \left| \sum_{i=1}^{N_\nu} X_i \right| \right) \approx \sqrt{N_\nu}$$

by the central limit theorem.

Square root law by Penrose.
Result

$$E_{\mu} \left( \left| \sum_{i=1}^{N_\nu} X_i \right| \right) \simeq \int \left| z \right| d\mu(z) \ N_\nu$$
The space \( \{-1, +1\}^N \) with the probability measure \( \mathbb{P} \) can be looked upon as a spin system which is invariant under spin flips. Let \( Q^{(N)} \) denote the uniform distribution on \( \Omega = \{-1, +1\}^N \). \( Q^{(N)} \) corresponds to independent, identically distributed (and uniformly distributed) \( X_i \).

Any probability measure \( \mathbb{P} \) on \( \Omega \) which is spin-flip invariant can be written as

\[
\mathbb{P} = Z^{-1} e^{\beta H(x_1, \ldots, x_N)} Q^{(N)}
\]

where \( Z \) is a normalizing constant

and \( H(x_1, \ldots, x_N) = H(-x_1, \ldots, -x_N) \)

where \( \beta \) plays the role of an inverse temperature (or a coupling strength).
• \( H(X_1, \ldots, X_N) \equiv 1 \) gives the model with independent voters.
Examples

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- The Common Believe Model can be interpreted as a system of noninteracting spins in a random, but constant magnetic field.
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- $H(X_1, \ldots, X_N) \equiv 1$ gives the model with independent voters.
- The Common Believe Model can be interpreted as a system of noninteracting spins in a random, but constant magnetic field.
- The Curie-Weiss Model is given by the Hamiltonian

$$H_{CW}(x_1, \ldots, x_N) = -\frac{1}{2N} \left( \sum_{i=1}^{N} X_i \right)^2 .$$

In this model the spins (voters) tend to behave in the same way as (the average of) the others. If the parameter $\beta$ is very small, the spins are almost independent, if $\beta$ is very large almost all of the spins are aligned.
For $\beta < 1$ we have

$$\mathbb{E}_{\beta}^{CW} \left( \left| \sum_{i=1}^{N} X_i \right| \right) \approx \sqrt{N}$$

For $\beta > 1$ we have

$$\mathbb{E}_{\beta}^{CW} \left( \left| \sum_{i=1}^{N_{\nu}} X_i \right| \right) \approx N$$
Optimal Weights for the Curie-Weiss Model

**Results**

- For $\beta < 1$ we have
  \[
  \mathbb{E}_\beta^{CW} \left( \left| \sum_{i=1}^{N} X_i \right| \right) \approx \sqrt{N}
  \]

- For $\beta > 1$ we have
  \[
  \mathbb{E}_\beta^{CW} \left( \left| \sum_{i=1}^{N_\nu} X_i \right| \right) \approx N
  \]

- For $\beta = 1$ we have
  \[
  \mathbb{E}_\beta^{CW} \left( \left| \sum_{i=1}^{N_\nu} X_i \right| \right) \approx N^{\frac{3}{4}}
  \]
In the **EU** with 27 member states there exist currently two voting systems for the **Council of Ministers**

- The voting system of **Nice** which was decided in Nice in 2000. The system is currently used.
- The voting system of **Lisbon** which was negotiated in Lisbon in 2007 and ratified last year. It will be effective in 2014 or so.
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The Polish government tried to block the Lisbon treaty because they believed that according to this treaty Poland is under-represented in the Council.

In fact, their slogan was:

**Square root or death.**
<table>
<thead>
<tr>
<th>State</th>
<th>Nice</th>
<th>Lisbon</th>
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<tr>
<td>Germany</td>
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<td>Sweden</td>
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<td>Luxembourg</td>
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<td>126,34</td>
</tr>
<tr>
<td>Malta</td>
<td>42,30</td>
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</tr>
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Definition

Let \((V, \mathcal{W})\) be a voting system and \(P\) be a (voting) measure on \((-1, 1)^V\). We may assume that \(V = \{1, 2, \ldots, N\}\) and identify \((-1, 1)^V\), \((-1, 1)^N\) and \(P(V)\).

Example: Parliament

If the voting rule is simple majority and \(N\) is odd, then \(\eta = \frac{1}{2}\) independent of the choice of the measure \(P\).
Efficiency

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The efficiency \(\eta\) of the voting system is defined as

\[ \eta := P(\mathcal{W}) \]
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The efficiency \(\eta\) of the voting system is defined as

\[
\eta := \mathbb{P}(\mathcal{W})
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Example: Parliament

If the voting rule is simple majority and \(N\) is odd, then \(\eta = \frac{1}{2}\) independent of the choice of the measure \(\mathbb{P}\).
The meaning of efficiency

If the efficiency of a voting system is close to $\frac{1}{2}$ it is relatively easy to make decisions. In almost any case either the supporters of a proposal or the supporters of the counter-proposal can enforce a decision in their respective favor. If the efficiency is low a stalemate will happen rather frequently.
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On the other hand, sometimes a high efficiency is not always desired. For example, in most countries an amendment of the constitution requires a large majority resulting in low efficiency.

For example in the European Union EU the member states tend to be afraid of being overruled by the others. So, they prefer to keep the efficiency low, thus strengthening their blocking power.
We are interested in the dependence of $\eta$ on the number $N$ of voters.

This plays a crucial role in enlargement processes. For example the EU was extended various times, in particular in 2004 and 2007 from 15 members to 27 members. The voting mechanism is (a combination of instances of) weighted voting with a quota kept fixed at about 70 %. It look like the politicians believed the kept also the efficiency of the system fixed. Is this true?
Setting

Let $w_i$ be an infinite sequence of weights, $q$ a quota and consider the voting systems $V_N = \{1, 2, \ldots, N\}$ with voting weights $w_1, w_2, \ldots, w_N$ and quota $q$.

Let $\mathbb{P}$ denote the product measure on $V_N$ and set

$$a_N := \frac{\sum_{i=1}^{N} w_i^2}{\left( \sum_{i=1}^{N} w_i \right)^2} \to 0$$
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**Result**

Under these conditions

$$\eta_N := \mathbb{P}(\mathcal{W}_N) = \mathcal{O}(e^{-Ca_N})$$
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Result

Under these conditions

$$\eta_N := \mathbb{P}(W_N) = \simeq \mu([q, 1]) + \frac{1}{2} \mu(\{q\})$$