Using Institutions for the Study of Qualitative and Quantitative Conditional Logics

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Abstract. It is well known that conditionals need a non-classical environment to be evaluated. In this paper, we present a formalization of conditional logic in the framework of institutions. In regarding both qualitative and probabilistic conditional logic as abstract logical systems, we investigate how they can be related to one another, on the one hand, and to the institution of propositional logic, on the other hand. In spite of substantial differences between these three logics, we find surprisingly clear formal relationships between them.

1 Introduction

In [6], Goguen and Burstall introduced institutions as a general framework for logical systems. An institution formalizes the informal notion of a logical system, including syntax, semantics, and the relation of satisfaction between them. The latter poses the major requirement for an institution: that the satisfaction relation is consistent under the change of notation. Institutions have also been used as a basis for specification and development languages, see e.g. [2, 13, 7].

While the examples for institutions in [6] and [7] are based on classical logic, in [1] it is shown that also probabilistic logic can be formalized as an institution. In this paper, we will apply the theory of institutions to conditionals, investigating how the logics of qualitative and probabilistic conditionals fit into that framework. As default rules, conditionals have played a major role in defeasible reasoning (see, e.g., [9]), and assigning probabilities to them opens up a whole universe of possibilities to quantify the (un)certainty of information. It is well-known that conditionals are substantially different from classical logical entities [5]. Nevertheless, we will show that their logic can be formalized as an institution, and thus be compared to classical logical institutions, e.g. that of propositional logic. In detail, we will define an institution of conditional logic, using the semantics of plausibility preorders going back to the works of Lewis [10] and Stalnaker [14], as well as an institution of probabilistic conditionals, using the usual semantics of conditional probabilities.

Now that the logic of conditionals has been given an abstract formal frame, we can also study its relationships to other logics, also being viewed as institutions.

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To do so, the formal tool of *institution morphisms* [6, 7] can be used. This will tell us precisely the possibilities how we can interpret e.g. probabilistic conditionals as qualitative conditionals, or in a propositional setting, and vice versa. We will prove that indeed, the institutions of propositional logic, conditional logic and probabilistic conditional logic can be related intuitively, but we will also find that there is essentially exactly one such connection between each pair of these logics. In particular, we will necessarily arrive at a three-valued interpretation of probabilistic conditionals in the propositional framework, reminding us of the three-valued semantics of conditionals [5].

In Sec. 2, we formalize conditional logic as an institution. In Sec. 3, we investigate in detail the relationships between the institutions of conditional logic and of propositional, probabilistic, and probabilistic conditional logic. Section 4 contains some conclusions and points out further work.

2 Institutions and the Logic of Conditionals

After recalling the definition of an institution and fixing some basic notation, we first present propositional logic in the institution framework. We then formalize in three steps probabilistic propositional logic, the logic of probabilistic conditionals, and the logic of (qualitative) conditionals as institutions.

2.1 Preliminaries: Basic Definitions and Notations

If C is a category, $|C|$ denotes the objects of C and $/C/$ its morphisms; for both objects $c \in |C|$ and morphisms $\varphi \in /C/$, we also write just $c \in C$ and $\varphi \in C$, respectively. C^{op} is the opposite category of C, with the direction of all morphisms reversed. The composition of two functors $F : C \to C'$ and $G : C' \to C''$ is denoted by $G \circ F$ (first apply F, then G). For functors $F, G : C \to C'$, a *natural transformation* η from F to G, denoted by $\eta : F \Longrightarrow G$, assigns to each object $c \in |C|$ a morphism $\eta_c : F(C) \to G(C) \in /C'/$ such that for every morphism $\varphi : c \to d \in /C/$ we have $\eta_d \circ F(\varphi) = G(\varphi) \circ \eta_c$. $\mathcal{SET}$ and $\mathcal{CAT}$ denote the categories of sets and of categories, respectively. (For more information about categories, see e.g. [8] or [11].) The central institution definition is the following:

Definition 1. [6] *An institution is a quadruple* $Inst = \langle\, Sig, Mod, Sen, \models \rangle$ *with a category* Sig *of signatures as objects, a functor* $Mod : Sig \to \mathcal{CAT}^{op}$ *yielding the category of* Σ*-models for each signature* Σ*, a functor* $Sen : Sig \to \mathcal{SET}$ *yielding the sentences over a signature, and a* $|Sig|$*-indexed relation* $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ *such that for each signature morphism* $\varphi : \Sigma \to \Sigma' \in /Sig/$*, for each* $m' \in |Mod(\Sigma')|$*, and for each* $f \in Sen(\Sigma)$ *the following* satisfaction condition *holds:* $m' \models_{\Sigma'} Sen(\varphi)(f)$ *iff* $Mod(\varphi)(m') \models_\Sigma f$*.*

For sets F, G of Σ-sentences and a Σ-model m we write $m \models_\Sigma F$ iff $m \models_\Sigma f$ for all $f \in F$. The satisfaction relation is lifted to semantical entailment $\models_\Sigma$ between sentences by defining $F \models_\Sigma G$ iff for all Σ-models m with $m \models_\Sigma F$ we have $m \models_\Sigma G$. $F^\bullet = \{f \in Sen(\Sigma) \mid F \models_\Sigma f\}$ is called the *closure* of F, and F

is *closed* if $F = F^\bullet$. The closure operator fulfils the *closure lemma* $\varphi(F^\bullet) \subseteq \varphi(F)^\bullet$ and various other nice properties like $\varphi(F^\bullet)^\bullet = \varphi(F)^\bullet$ or $(F^\bullet \cup G)^\bullet = (F \cup G)^\bullet$. A consequence of the closure lemma is that entailment is preserved under change of notation carried out by a signature morphism, i.e. $F \models_\Sigma G$ implies $\varphi(F) \models_{\varphi(\Sigma)} \varphi(G)$ (but not vice versa).

2.2 The Institution of Propositional Logic

In all circumstances, propositional logic seems to be the most basic logic. The components of its institution $Inst_\mathcal{B} = \langle\, Sig_\mathcal{B},\ Mod_\mathcal{B},\ Sen_\mathcal{B},\ \models_\mathcal{B} \,\rangle$ will be defined in the following.

Signatures: $Sig_\mathcal{B}$ is the category of propositional signatures. A propositional signature $\Sigma \in |\, Sig_\mathcal{B}\,|$ is a (finite) set of propositional variables, $\Sigma = \{a_1, a_2, \ldots\}$. A propositional signature morphism $\varphi : \Sigma \to \Sigma' \in /Sig_\mathcal{B}/$ is a function mapping propositional variables to propositional variables.

Models: For each signature $\Sigma \in Sig_\mathcal{B}$, $Mod_\mathcal{B}(\Sigma)$ contains the set of all propositional interpretations for Σ, i.e. $|\, Mod_\mathcal{B}(\Sigma)\,| = \{I \mid I : \Sigma \to Bool\}$ where $Bool = \{true, false\}$. Due to its simple structure, the only morphisms in $Mod_\mathcal{B}(\Sigma)$ are the identity morphisms. For each signature morphism $\varphi : \Sigma \to \Sigma' \in Sig_\mathcal{B}$, we define the morphism (i.e. the functor in $\mathcal{CAT}^{op}$) $Mod_\mathcal{B}(\varphi) : Mod_\mathcal{B}(\Sigma') \to Mod_\mathcal{B}(\Sigma)$ by $(Mod_\mathcal{B}(\varphi)(I'))(a_i) := I'(\varphi(a_i))$ where $I' \in Mod_\mathcal{B}(\Sigma')$ and $a_i \in \Sigma$.

Sentences: For each signature $\Sigma \in Sig_\mathcal{B}$, the set $Sen_\mathcal{B}(\Sigma)$ contains the usual propositional formulas constructed from the propositional variables in Σ and the logical connectives $\wedge$ (and), $\vee$ (or), and $\neg$ (not). Additionally, the classical (material) implication $A \Rightarrow B$ is used as a syntactic variant for $\neg A \vee B$. The symbols $\top$ and $\bot$ denote a tautology (like $a \vee \neg a$) and a contradiction (like $a \wedge \neg a$), respectively.

For each signature morphism $\varphi : \Sigma \to \Sigma' \in Sig_\mathcal{B}$, the function $Sen_\mathcal{B}(\varphi) : Sen_\mathcal{B}(\Sigma) \to Sen_\mathcal{B}(\Sigma')$ is defined by straightforward inductive extension on the structure of the formulas; e.g., $Sen_\mathcal{B}(\varphi)(a_i) = \varphi(a_i)$ and $Sen_\mathcal{B}(\varphi)(A \wedge B) = Sen_\mathcal{B}(\varphi)(A) \wedge Sen_\mathcal{B}(\varphi)(B)$. In the following, we will abbreviate $Sen_\mathcal{B}(\varphi)(A)$ by just writing $\varphi(A)$. In order to simplify notations, we will often replace conjunction by juxtaposition and indicate negation of a formula by barring it, i.e. $AB = A \wedge B$ and $\overline{A} = \neg A$. An *atomic formula* is a formula consisting of just a propositional variable, a *literal* is a positive or a negated atomic formula, an *elementary conjunction* is a conjunction of literals, and a *complete conjunction* is an elementary conjunction where all atomic formulas appear once, either in positive or in negated form. Ω_Σ denotes the set of all complete conjunctions over a signature Σ; if Σ is clear from the context, we may drop the index Σ. Note that there is an obvious bijection between $|\, Mod_\mathcal{B}(\Sigma)\,|$ and Ω_Σ, associating with $I \in |\, Mod_\mathcal{B}(\Sigma)\,|$ the complete conjunction $\omega_I \in \Omega_\Sigma$ in which an atomic formula $a_i \in \Sigma$ occurs in positive form iff $I(a_i) = true$.

Satisfaction relation: For any $\Sigma \in |\, Sig_\mathcal{B}\,|$, the satisfaction relation $\models_{\mathcal{B},\Sigma} \ \subseteq |\, Mod_\mathcal{B}(\Sigma)\,| \times Sen_\mathcal{B}(\Sigma)$ is defined as expected for propositional logic, e.g.

$I \models_{\mathcal{B},\Sigma} a_i$ iff $I(a_i) = true$ and $I \models_{\mathcal{B},\Sigma} A \wedge B$ iff $I \models_{\mathcal{B},\Sigma} A$ and $I \models_{\mathcal{B},\Sigma} B$ for $a_i \in \Sigma$ and $A, B \in Sen_{\mathcal{B}}(\Sigma)$.

Proposition 1. *$Inst_{\mathcal{B}} = \langle\, Sig_{\mathcal{B}},\ Mod_{\mathcal{B}},\ Sen_{\mathcal{B}},\ \models_{\mathcal{B}}\,\rangle$ is an institution.*

Example 1. Let $\Sigma = \{s, t, u\}$ and $\Sigma' = \{a, b, c\}$ be two propositional signatures with the atomic propositions s – *being a scholar*, t – *being not married*, u – *being single* and a – *being a student*, b – *being young*, c – *being unmarried*. Let I' be the Σ'-model with $I'(a) = true$, $I'(b) = true$, $I'(c) = false$. Let $\varphi : \Sigma \to \Sigma' \in Sig_{\mathcal{B}}$ be the signature morphism with $\varphi(s) = a$, $\varphi(t) = c$, $\varphi(u) = c$. The functor $Mod_{\mathcal{B}}(\varphi)$ takes I' to the Σ-model $I := Mod_{\mathcal{B}}(\varphi)(I')$, yielding $I(s) = I'(a) = true$, $I(t) = I'(c) = false$, $I(u) = I'(c) = false$.

2.3 The Institution of Probabilistic Propositional Logic

Based on $Inst_{\mathcal{B}}$, we can now define the institution of probabilistic propositional logic $Inst_{\mathcal{P}} = \langle\, Sig_{\mathcal{P}},\ Mod_{\mathcal{P}},\ Sen_{\mathcal{P}},\ \models_{\mathcal{P}}\,\rangle$. We will first give a very short introduction to probabilistics as far as it is needed here.

Let $\Sigma \in |\,Sig_{\mathcal{B}}\,|$ be a propositional signature. A *probability distribution* (or *probability function*) *over* Σ is a function $P : Sen_{\mathcal{B}}(\Sigma) \to [0, 1]$ such that $P(\top) = 1$, $P(\bot) = 0$, and $P(A \vee B) = P(A) + P(B)$ for any formulas $A, B \in Sen_{\mathcal{B}}(\Sigma)$ with $AB = \bot$. Each probability distribution P is determined uniquely by its values on the complete conjunctions $\omega \in \Omega_\Sigma$, since $P(A) = \sum_{\omega \in \Omega_\Sigma, \omega \models_{\mathcal{B},\Sigma} A} P(\omega)$. For two propositional formulas $A, B \in Sen_{\mathcal{B}}(\Sigma)$ with $P(A) > 0$, the *conditional probability of B given A*, denoted by $P(B|A)$, is $\dfrac{P(AB)}{P(A)}$. Any subset $\Sigma_1 \subseteq \Sigma$ gives rise to a distribution $P_{\Sigma_1} : Sen_{\mathcal{B}}(\Sigma_1) \to [0, 1]$ by virtue of defining $P_{\Sigma_1}(\omega_1) = \sum_{\omega \in \Omega_\Sigma, \omega \models_{\mathcal{B},\Sigma} \omega_1} P(\omega)$ for all $\omega_1 \in \Omega_{\Sigma_1}$; P_{Σ_1} is called the *marginal distribution of P on Σ_1*.

Signatures: $Sig_{\mathcal{P}}$ is identical to the category of propositional signatures, i.e. $Sig_{\mathcal{P}} = Sig_{\mathcal{B}}$.

Models: For each signature Σ, the objects of $Mod_{\mathcal{P}}(\Sigma)$ are probability distributions over the propositional variables, i.e.

$$|\,Mod_{\mathcal{P}}(\Sigma)\,| \;=\; \{P \mid P \text{ is a probability distribution over } \Sigma\}$$

As for $Mod_{\mathcal{B}}(\Sigma)$, we assume in this paper that the only morphisms in $Mod_{\mathcal{P}}(\Sigma)$ are the identity morphisms.

For each signature morphism $\varphi : \Sigma \to \Sigma'$, we define a functor $Mod_{\mathcal{P}}(\varphi) : Mod_{\mathcal{P}}(\Sigma') \to Mod_{\mathcal{P}}(\Sigma)$ by mapping each distribution P' over Σ' to a distribution $Mod_{\mathcal{P}}(\varphi)(P')$ over Σ. $Mod_{\mathcal{P}}(\varphi)(P')$ is defined by giving its value for all complete conjunctions over Σ:

$$(Mod_{\mathcal{P}}(\varphi)(P'))(\omega) \;:=\; P'(\varphi(\omega)) = \sum_{\omega' \models_{\mathcal{B},\Sigma'} \varphi(\omega)} P'(\omega')$$

where ω and ω' are complete conjunctions over Σ and Σ', respectively.

Sentences: For each signature Σ, the set $Sen_{\mathcal{P}}(\Sigma)$ contains *probabilistic facts* of the form $A[x]$ where $A \in Sen_{\mathcal{B}}(\Sigma)$ is a propositional formula from $Inst_{\mathcal{B}}$. $x \in [0,1]$ is a probability value indicating the degree of certainty for the occurrence of A.

For each signature morphism $\varphi : \Sigma \to \Sigma'$, the extension $Sen_{\mathcal{P}}(\varphi) : Sen_{\mathcal{P}}(\Sigma) \to Sen_{\mathcal{P}}(\Sigma')$ is defined by $Sen_{\mathcal{P}}(\varphi)(A[x]) = \varphi(A)[x]$.

Satisfaction relation: The satisfaction relation $\models_{\mathcal{P},\Sigma} \subseteq |Mod_{\mathcal{P}}(\Sigma)| \times Sen_{\mathcal{P}}(\Sigma)$ is defined, for any $\Sigma \in |Sig_{\mathcal{P}}|$, by

$$P \models_{\mathcal{P},\Sigma} A[x] \quad \text{iff} \quad P(A) = x$$

Note that, since $P(\overline{A}) = 1 - P(A)$ for each formula $A \in Sen_{\mathcal{B}}(\Sigma)$, it holds that $P \models_{\mathcal{P},\Sigma} A[x]$ iff $P \models_{\mathcal{P},\Sigma} \overline{A}[1-x]$.

Proposition 2. *$Inst_{\mathcal{P}} = \langle Sig_{\mathcal{P}}, Mod_{\mathcal{P}}, Sen_{\mathcal{P}}, \models_{\mathcal{P}} \rangle$ is an institution.*

2.4 The Institution of Probabilistic Conditional Logic

We now use $Inst_{\mathcal{P}}$ to define the institution of probabilistic conditionals $Inst_{\mathcal{C}} = \langle Sig_{\mathcal{C}}, Mod_{\mathcal{C}}, Sen_{\mathcal{C}}, \models_{\mathcal{C}} \rangle$.

Signatures: $Sig_{\mathcal{C}}$ is identical to the category of propositional signatures, i.e. $Sig_{\mathcal{C}} = Sig_{\mathcal{P}} = Sig_{\mathcal{B}}$.

Models: The models for probabilistic conditional logic are again probability distributions over the propositional variables. Therefore, the model functor can be taken directly from probabilistic propositional logic, giving us $Mod_{\mathcal{C}} = Mod_{\mathcal{P}}$.

Sentences: For each signature Σ, the set $Sen_{\mathcal{C}}(\Sigma)$ contains *probabilistic conditionals* (sometimes also called *probabilistic rules*) of the form $(B|A)[x]$ where $A, B \in Sen_{\mathcal{B}}(\Sigma)$ are propositional formulas from $Inst_{\mathcal{B}}$. $x \in [0,1]$ is a probability value indicating the degree of certainty for the occurrence of B under the condition A. – Note that the sentences from $Inst_{\mathcal{P}}$ are included implicitly since a probabilistic fact of the form $B[x]$ can easily be expressed as a conditional $(B|\top)[x]$ with a tautology as trivial antecedent.

For each signature morphism $\varphi : \Sigma \to \Sigma'$, the extension $Sen_{\mathcal{C}}(\varphi) : Sen_{\mathcal{C}}(\Sigma) \to Sen_{\mathcal{C}}(\Sigma')$ is defined by straightforward inductive extension on the structure of the formulas: $Sen_{\mathcal{C}}(\varphi)((B|A)[x]) = (\varphi(B)|\varphi(A))[x]$.

Satisfaction relation: The satisfaction relation $\models_{\mathcal{C},\Sigma} \subseteq |Mod_{\mathcal{C}}(\Sigma)| \times Sen_{\mathcal{C}}(\Sigma)$ is defined, for any $\Sigma \in |Sig_{\mathcal{C}}|$, by

$$P \models_{\mathcal{C},\Sigma} (B|A)[x] \quad \text{iff} \quad P(A) > 0 \text{ and } P(B \mid A) = \frac{P(AB)}{P(A)} = x$$

Note that for probabilistic facts we have $P \models_{\mathcal{C},\Sigma} (B|\top)[x]$ iff $P(B) = x$ from the definition of the satisfaction relation since $P(\top) = 1$. Thus, $P \models_{\mathcal{P},\Sigma} B[x]$ iff $P \models_{\mathcal{C},\Sigma} (B|\top)[x]$.

Proposition 3. *$Inst_{\mathcal{C}} = \langle Sig_{\mathcal{C}}, Mod_{\mathcal{C}}, Sen_{\mathcal{C}}, \models_{\mathcal{C}} \rangle$ is an institution.*

2.5 The Institution of Conditional Logic

The institution of conditional logic is $\mathit{Inst}_{\mathcal{K}} = \langle\, \mathit{Sig}_{\mathcal{K}},\ \mathit{Mod}_{\mathcal{K}},\ \mathit{Sen}_{\mathcal{K}},\ \models_{\mathcal{K}}\,\rangle$ with:

Signatures: $\mathit{Sig}_{\mathcal{K}}$ is again identical to the category of propositional signatures, i.e. $\mathit{Sig}_{\mathcal{K}} = \mathit{Sig}_{\mathcal{C}} = \mathit{Sig}_{\mathcal{P}} = \mathit{Sig}_{\mathcal{B}}$.

Models: Various types of models have been proposed to interpret conditionals adequately within a formal system (cf. e.g. [12]). Many of them are based on considering possible worlds which can be thought of as being represented by classical logical interpretations $|\,\mathit{Mod}_{\mathcal{B}}(\Sigma)\,|$, or complete conjunctions $\omega \in \Omega$ (as defined in Sec. 2.2), respectively. One of the most prominent approaches is the *system-of-spheres* model of Lewis [10] which makes use of a notion of similarity between possible worlds. This idea of comparing worlds and evaluating conditionals with respect to the "nearest" or "best" worlds (which are somehow selected) is common to very many approaches in conditional logics. So, in order to base our conditional logic on quite a general semantics, we take the models to be total preorders over classical propositional interpretations, i.e.

$$|\,\mathit{Mod}_{\mathcal{K}}(\Sigma)\,| \;=\; \{R \;\mid\; R \text{ is a total preorder on } |\,\mathit{Mod}_{\mathcal{B}}(\Sigma)\,|\}$$

where a total preorder R is a reflexive and transitive relation such that for any two elements I_1, I_2, we have $(I_1, I_2) \in R$ or $(I_2, I_1) \in R$ (possibly both).

By identifying $\mathit{Mod}_{\mathcal{B}}(\Sigma)$ with the set of possible worlds Ω, we will consider the models $R \in \mathit{Mod}_{\mathcal{K}}(\Sigma)$ to be total preorders on Ω, ordering the possible worlds according to their *plausibility*. By convention, the least worlds are the most plausible worlds. We will also use the infix notation $\omega_1 \preceq_R \omega_2$ instead of $(\omega_1, \omega_2) \in R$. As usual, we introduce the $\prec_R$-relation by saying that $\omega_1 \prec_R \omega_2$ iff $\omega_1 \preceq_R \omega_2$ and not $\omega_2 \preceq_R \omega_1$. Furthermore, $\omega_1 \approx_R \omega_2$ means that both $\omega_1 \preceq_R \omega_2$ and $\omega_2 \preceq_R \omega_1$ hold.

Each $R \in \mathit{Mod}_{\mathcal{K}}(\Sigma)$ induces a partitioning $\Omega_0, \Omega_1, \ldots$ of Ω, such that all worlds in the same partitioning subset are considered equally plausible ($\omega_1 \approx_R \omega_2$ for $\omega_1, \omega_2 \in \Omega_j$), and whenever $\omega_1 \in \Omega_i$ and $\omega_2 \in \Omega_k$ with $i < k$, then $\omega_1 \prec_R \omega_2$. Let $\mathit{Min}(R)$ denote the set of R-minimal worlds in Ω, i.e.

$$\mathit{Min}(R) \;=\; \Omega_0 \;=\; \{\omega_0 \in \Omega \mid \omega_0 \preceq_R \omega \text{ for all } \omega \in \Omega\}$$

Each $R \in \mathit{Mod}_{\mathcal{K}}(\Sigma)$ induces a total preorder on $\mathit{Sen}_{\mathcal{B}}(\Sigma)$ by setting

$$\begin{array}{lll} A \preceq_R B & \text{iff} & \text{for all } \omega_2 \in \Omega \text{ with } \omega_2 \models_{\mathcal{B},\Sigma} B \\ & & \text{there exists } \omega_1 \in \Omega \text{ with } \omega_1 \models_{\mathcal{B},\Sigma} A \text{ such that } \omega_1 \preceq_R \omega_2 \end{array}$$

So, A is considered to be at least as plausible as B (with respect to R) iff the most plausible worlds satisfying A are at least as plausible as any world satisfying B. In particular, if $B \models_{\mathcal{B},\Sigma} A$, then $A \preceq_R B$ for each $R \in \mathit{Mod}_{\mathcal{K}}(\Sigma)$, since $\omega \models_{\mathcal{B},\Sigma} B$ implies $\omega \models_{\mathcal{B},\Sigma} A$. Again, $A \prec_R B$ means both $A \preceq_R B$ and not $B \preceq_R A$. Note that $A \prec_R \bot$ for all $A \not\equiv \bot$.

As before, we only consider the identity morphisms in $\mathit{Mod}_{\mathcal{K}}(\Sigma)$ for this paper.

For each signature morphism $\varphi : \Sigma \to \Sigma'$, we define a functor $Mod_{\mathcal{K}}(\varphi) : Mod_{\mathcal{K}}(\Sigma') \to Mod_{\mathcal{K}}(\Sigma)$ by mapping a (total) preorder R' over $Mod_{\mathcal{B}}(\Sigma')$ to a (total) preorder $Mod_{\mathcal{K}}(\varphi)(R')$ over $Mod_{\mathcal{B}}(\Sigma)$ in the following way:

$$\omega_1 \preceq_{Mod_{\mathcal{K}}(\varphi)(R')} \omega_2 \quad \text{iff} \quad \varphi(\omega_1) \preceq_{R'} \varphi(\omega_2) \tag{1}$$

Note that on the left hand side of (1) the complete conjunctions ω_1 and ω_2 are viewed as models in $Mod_{\mathcal{B}}(\Sigma)$, whereas on the right hand side they are sentences in $Sen_{\mathcal{B}}(\Sigma)$.

It is straightforward to check that $Mod_{\mathcal{K}}(\varphi)(R')$ is a total preorder (the corresponding properties are all directly inherited by R'), so indeed $Mod_{\mathcal{K}}(\varphi)(R') \in Mod_{\mathcal{K}}(\Sigma)$. The connection between R' and $Mod_{\mathcal{K}}(\varphi)(R')$ defined by (1) can also be shown to hold for propositional sentences instead of worlds:

Lemma 1. *Let $A, B \in Sen_{\mathcal{B}}(\Sigma)$. Then $A \preceq_{Mod_{\mathcal{K}}(\varphi)(R')} B$ iff $\varphi(A) \preceq_{R'} \varphi(B)$.*

Corollary 1. *Let $A, B \in Sen_{\mathcal{B}}(\Sigma)$. Then $A \prec_{Mod_{\mathcal{K}}(\varphi)(R')} B$ iff $\varphi(A) \prec_{R'} \varphi(B)$.*

Sentences: For each signature Σ, the set $Sen_{\mathcal{K}}(\Sigma)$ contains (propositional) *conditionals* of the form $(B|A)$ where $A, B \in Sen_{\mathcal{B}}(\Sigma)$ are propositional formulas from $Inst_{\mathcal{B}}$. For $\varphi : \Sigma \to \Sigma'$, the extension $Sen_{\mathcal{K}}(\varphi)$ is defined as usual by $Sen_{\mathcal{K}}(\varphi)((B|A)) = (\varphi(B)|\varphi(A))$.

Satisfaction relation: The satisfaction relation $\models_{\mathcal{K},\Sigma} \;\subseteq\; |\, Mod_{\mathcal{K}}(\Sigma)| \times Sen_{\mathcal{K}}(\Sigma)$ is defined, for any $\Sigma \in |\, Sig_{\mathcal{K}}|$, by

$$R \models_{\mathcal{K},\Sigma} (B|A) \quad \text{iff} \quad AB \prec_R A\overline{B}$$

Therefore, a conditional $(B|A)$ is satisfied (or accepted) by the plausibility preorder R iff its confirmation AB is more plausible than its refutation $A\overline{B}$.

Proposition 4. *$Inst_{\mathcal{K}} = \langle\, Sig_{\mathcal{K}},\, Mod_{\mathcal{K}},\, Sen_{\mathcal{K}},\, \models_{\mathcal{K}} \,\rangle$ is an institution.*

Example 2. We continue our student-example in this qualitative conditional environment, so let Σ, Σ', φ be as defined in Example 1. Let R' be the following total preorder on Ω':

$$R' : \quad \overline{a}\overline{b}\overline{c} \prec_{R'} abc \approx_{R'} \overline{a}bc \prec_{R'} ab\overline{c} \approx_{R'} a\overline{b}c \approx_{R'} a\overline{b}\overline{c} \approx_{R'} \overline{a}b\overline{c} \approx_{R'} \overline{a}\overline{b}c$$

Now, for instance $R' \models_{\mathcal{K},\Sigma'} (a|\top)$ since $\top\overline{a} \equiv \overline{a}$, $\top\overline{\overline{a}} \equiv a$, and $\overline{a} \prec_{R'} a$. Thus, under R', it is more plausible to be not a student than to be a student. Furthermore, $R' \models_{\mathcal{K},\Sigma'} (c|a)$ – *students* are supposed to be *unmarried* since under R', ac is more plausible than $a\overline{c}$.

Under $Mod_{\mathcal{K}}(\varphi)$, R' is mapped onto $R = Mod_{\mathcal{K}}(\varphi)(R')$ where R is the following total preorder on Ω:

$$R : \quad \overline{s}\overline{t}\overline{u} \prec_R \overline{s}tu \approx_R stu \prec_R s\overline{t}\overline{u} \prec_R st\overline{u} \approx_R s\overline{t}u \approx_R \overline{s}t\overline{u} \approx_R \overline{s}\overline{t}u$$

As expected, the conditionals $(t|s)$ and $(u|s)$, both corresponding to $(c|a)$ in $Sen_{\mathcal{K}}(\Sigma')$ under φ, are satisfied by R – here, *scholars* are supposed to be both *not married* and *single*.

3 Relating Conditional Logic to Other Logics

Having stepwise developed conditional logic, we now turn to study its interrelationships to the other logics. For instance, there is an obvious translation of sentences mapping A to $(A|\top)$ and mapping $(A|B)$ to $(A|B)[1]$. Furthermore, there is a similar obvious transformation of a propositional interpretation I to a conditional logic model viewing I to be more plausible than any other interpretation, which in turn are considered to be all equally plausible. What happens to satisfaction and entailment when using such translations? In order to make these questions more precise, we use the notion of institution morphisms introduced in [6] (see also [7]).

An institution morphism Φ expresses a relation between two institutions *Inst* und *Inst'* such that the satisfaction condition of *Inst* may be computed by the satisfaction condition of *Inst'* if we translate it according to Φ. The translation is done by relating every *Inst*-signature Σ to an *Inst'*-signature Σ', each Σ'-sentence to a Σ-sentence, and each Σ-model to a Σ'-model.

Definition 2. [6] *Let* $Inst = \langle\, Sig, Mod, Sen, \models \,\rangle$ *and* $Inst' = \langle\, Sig', Mod', Sen', \models' \,\rangle$ *be two institutions. An* institution morphism Φ *from* $Inst$ *to* $Inst'$ *is a triple* $\langle\, \phi, \alpha, \beta \,\rangle$ *with a functor* $\phi : Sig \to Sig'$*, a natural transformation* $\alpha : Sen' \circ \phi \Longrightarrow Sen$*, and a natural transformation* $\beta : Mod \Longrightarrow Mod' \circ \phi$ *such that for each* $\Sigma \in |\,Sig\,|$*, for each* $m \in |\,Mod(\Sigma)\,|$*, and for each* $f' \in Sen'(\phi(\Sigma))$ *the following* satisfaction condition (for institution morphisms) *holds:* $\quad m \models_\Sigma \alpha_\Sigma(f') \quad$ *iff* $\quad \beta_\Sigma(m) \models'_{\phi(\Sigma)} f'$.

Since $Inst_\mathcal{B}$, $Inst_\mathcal{P}$, $Inst_\mathcal{C}$, and $Inst_\mathcal{K}$ all have the same category $Sig_\mathcal{B}$ of signatures, a natural choice for the signature translation component ϕ in any morphism between these institutions is the identity $id_{Sig_\mathcal{B}}$ which we will use in the following.

3.1 Relating Propositional and Conditional Logic

The sentences of $Inst_\mathcal{B}$ and $Inst_\mathcal{K}$ can be related in an intuitive way by sending a propositional formula A to the conditional $(A|\top)$ having the trivial antecedent $\top$. It is easy to check that this translation yields a natural transformations

$$\alpha_{\mathcal{B}/\mathcal{K}} : Sen_\mathcal{B} \Longrightarrow Sen_\mathcal{K} \qquad \alpha_{\mathcal{B}/\mathcal{K},\Sigma}(A) = (A|\top)$$

Similarly, there is also an intuitive way of mapping a propositional model I to a conditional logic model (which we will denote by R_I). This model R_I views I to be more plausible than any other world, and all other worlds are looked upon as equally plausible. With ω_I denoting the unique complete conjunction with $I \models_{\mathcal{B},\Sigma} \omega_I$, the preorder R_I thus partitions Ω into the two sets $\{\omega_I\}$ and $\Omega\backslash\{\omega_I\}$. Therfore, $\omega_I \prec_{R_I} \omega$ for all $\omega \neq \omega_I$. Formally, R_I is defined by

$$\omega_1 \preceq_{R_I} \omega_2 \quad \text{iff} \quad I \models_{\mathcal{B},\Sigma} \omega_1 \text{ or } (I \not\models_{\mathcal{B},\Sigma} \omega_1 \text{ and } I \not\models_{\mathcal{B},\Sigma} \omega_2)$$

It is straightforward to check that this yields a natural transfromation

$$\beta_{\mathcal{B}/\mathcal{K}} : Mod_\mathcal{B} \Longrightarrow Mod_\mathcal{K} \qquad \beta_{\mathcal{B}/\mathcal{K},\Sigma}(I) = R_I$$

Having identified obvious standard translations for sentences and models from $\mathit{Inst}_{\mathcal{B}}$ to $\mathit{Inst}_{\mathcal{K}}$, the next question is how to use them in relations between these two institutions. As intuitive as the sentence translation $\alpha_{\mathcal{B}/\mathcal{K}}$ appears, the next proposition shows that it can not be used to define an institution morphism from $\mathit{Inst}_{\mathcal{K}}$ to $\mathit{Inst}_{\mathcal{B}}$:

Proposition 5. *There is no β such that $\langle\, id_{\mathit{Sig}_{\mathcal{B}}},\, \alpha_{\mathcal{B}/\mathcal{K}},\, \beta\,\rangle : \mathit{Inst}_{\mathcal{K}} \longrightarrow \mathit{Inst}_{\mathcal{B}}$ is an institution morphism.*

When going in the other direction from $\mathit{Inst}_{\mathcal{B}}$ to $\mathit{Inst}_{\mathcal{K}}$ using the model translation $\beta_{\mathcal{B}/\mathcal{K}} : \mathit{Mod}_{\mathcal{B}} \Longrightarrow \mathit{Mod}_{\mathcal{K}}$, we must map conditionals to propositional formulas. A possible choice is to map $(B|A)$ to the formula AB confirming the conditional, thereby yielding the natural transformation

$$\alpha_{\mathcal{K}/\mathcal{B}} : \mathit{Sen}_{\mathcal{K}} \Longrightarrow \mathit{Sen}_{\mathcal{B}} \qquad \text{with} \qquad \alpha_{\mathcal{K}/\mathcal{B},\Sigma}((B|A)) = AB$$

The next proposition shows that $\beta_{\mathcal{B}/\mathcal{K}}$ gives rise to exactly one institution morphism from $\mathit{Inst}_{\mathcal{B}}$ to $\mathit{Inst}_{\mathcal{K}}$, namely the one using $\alpha_{\mathcal{K}/\mathcal{B}}$ for sentence translation:

Proposition 6. *$\langle\, id_{\mathit{Sig}_{\mathcal{B}}},\, \alpha,\, \beta_{\mathcal{B}/\mathcal{K}}\,\rangle : \mathit{Inst}_{\mathcal{B}} \longrightarrow \mathit{Inst}_{\mathcal{K}}$ is an institution morphism iff $\alpha = \alpha_{\mathcal{K}/\mathcal{B}}$.*

3.2 Relating Conditional and Probabilistic Conditional Logic

It is quite straightforward to relate the sentences of $\mathit{Inst}_{\mathcal{K}}$ and $\mathit{Inst}_{\mathcal{C}}$ by sending a conditional to a probabilistic conditional with trivial probability 1, yielding the natural transformation

$$\alpha_{\mathcal{K}/\mathcal{C}} : \mathit{Sen}_{\mathcal{K}} \Longrightarrow \mathit{Sen}_{\mathcal{C}} \qquad \text{with} \qquad \alpha_{\mathcal{K}/\mathcal{C},\Sigma}((B|A)) = (B|A)[1]$$

Relating the models of $\mathit{Inst}_{\mathcal{K}}$ and $\mathit{Inst}_{\mathcal{C}}$, however, is far less obvious. Many different ways can be devised to map preorders and probability distributions to one another. As a minimal requirement, we would certainly expect the preorder to be compatible with the ordering induced by the probabilities. As a first approach, we define a mapping sending a probability distribution P to a conditional logic model R_P. Under R_P, all complete conjunctions with a positive probability are considered most plausible, and all complete conjunctions with zero probability are taken as less (yet equally) plausible. Thus, R_P partitions Ω into two sets, namely $\mathit{Min}(R_P) = \{\omega \in \Omega \mid P(\omega) > 0\}$ and $\{\omega \in \Omega \mid P(\omega) = 0\}$. Formally, R_P is defined by

$$\omega_1 \preceq_{R_P} \omega_2 \quad \text{iff} \quad P(\omega_2) = 0 \text{ or } (P(\omega_1) > 0 \text{ and } P(\omega_2) > 0)$$

and it is easy to check that this yields a natural transformation

$$\beta_{\mathcal{C}/\mathcal{K}} : \mathit{Mod}_{\mathcal{C}} \Longrightarrow \mathit{Mod}_{\mathcal{K}} \qquad \text{with} \qquad \beta_{\mathcal{C}/\mathcal{K},\Sigma}(P) = R_P$$

The next proposition shows that this indeed gives rise to an institution morphism from $\mathit{Inst}_{\mathcal{C}}$ to $\mathit{Inst}_{\mathcal{K}}$ which involves $\alpha_{\mathcal{K}/\mathcal{C}}$.

Proposition 7. *$\langle\, id_{\mathit{Sig}_{\mathcal{B}}},\, \alpha_{\mathcal{K}/\mathcal{C}},\, \beta_{\mathcal{C}/\mathcal{K}}\,\rangle : \mathit{Inst}_{\mathcal{C}} \longrightarrow \mathit{Inst}_{\mathcal{K}}$ is an institution morphism.*

What other possibilities are there to generate a preorder from a probability distribution so that the intuitive sentence translation $\alpha_{\mathcal{K}/\mathcal{C}}$ yields an institution morphism? Although the preordering concept would allow a rather fine-grained hierarchy of plausibilities, it is surprising to see that the somewhat simplistic two-level approach of R_P is the *only* possibility to augment $\alpha_{\mathcal{K}/\mathcal{C}}$ towards an institution morphism.

Proposition 8. *If* $\langle\, id_{Sig_{\mathcal{B}}},\, \alpha_{\mathcal{K}/\mathcal{C}},\, \beta \,\rangle : Inst_{\mathcal{C}} \longrightarrow Inst_{\mathcal{K}}$ *is an institution morphism then* $\beta = \beta_{\mathcal{C}/\mathcal{K}}$.

Going in the other direction, i.e. from $Inst_{\mathcal{K}}$ to $Inst_{\mathcal{C}}$, we have to transform probabilistic conditionals into qualitative conditionals by a natural transformation $\alpha : Sen_{\mathcal{C}} \Longrightarrow Sen_{\mathcal{K}}$. We might anticipate problems in handling properly non-trivial probabilities, but we would certainly expect that $\alpha_{\Sigma}((B|A)[1]) = (B|A)$. The next proposition, however, shows even this to be impossible.

Proposition 9. *There is no institution morphism* $\langle\, id_{Sig_{\mathcal{B}}},\, \alpha,\, \beta \,\rangle : Inst_{\mathcal{K}} \longrightarrow Inst_{\mathcal{C}}$ *such that* $\alpha_{\Sigma}((B|A)[1]) = (B|A)$ *for all signatures* Σ.

3.3 Relating Propositional and Probabilistic Conditional Logic

What is the situation between propositional logic $Inst_{\mathcal{B}}$ and probabilistic conditional logic $Inst_{\mathcal{C}}$? Here, the obvious standard translations are

$$\alpha_{\mathcal{B}/\mathcal{C}} : Sen_{\mathcal{B}} \Longrightarrow Sen_{\mathcal{C}} \qquad \alpha_{\mathcal{B}/\mathcal{C},\Sigma}(A) = (A|\top)[1]$$

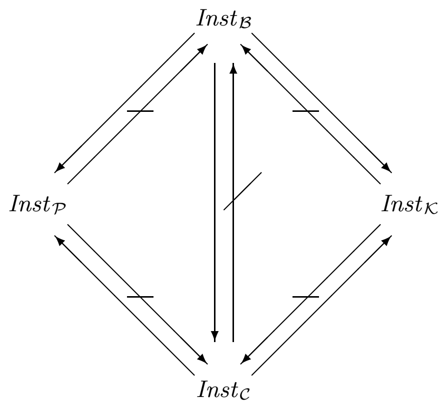
$$\beta_{\mathcal{B}/\mathcal{C}} : Mod_{\mathcal{B}} \Longrightarrow Mod_{\mathcal{C}} \qquad \beta_{\mathcal{B}/\mathcal{C},\Sigma}(I) = P_I \qquad P_I(\omega) = \begin{cases} 1 & \text{if } I(\omega) = \mathit{true} \\ 0 & \text{otherwise} \end{cases}$$

In [1] it is shown that no institution morphism using the sentence translation $\alpha_{\mathcal{B}/\mathcal{C}}$ exists. When going from propositional logic to probabilistic conditional logic using the model translation $\beta_{\mathcal{B}/\mathcal{C}}$, we have to map probabilistic conditionals to propositional sentences. However, possibly a little surpising at first sight, all probabilistic conditionals with probabilities other than 0 and 1 must be viewed as contradictory propositions.

Proposition 10. *If* $\langle\, id_{Sig_{\mathcal{B}}},\, \alpha,\, \beta_{\mathcal{B}/\mathcal{C}} \,\rangle : Inst_{\mathcal{B}} \longrightarrow Inst_{\mathcal{C}}$ *is an institution morphism, then for any* $\Sigma \in Sig_{\mathcal{B}}$*,* α_{Σ} *maps every sentence* $(B|A)[x]$ *with* $x \neq 0$ *and* $x \neq 1$ *to* $\perp$.

The only choices left for the translation of probabilistic conditionals is thus the translation of conditionals with the trivial probabilities 1 and 0. Although $(B|A)[1]$ represents a conditional *if A then B with probability 1*, we can not map it to the classical (material) implication $A \Rightarrow B$ (see [1]). By taking the antecedent as a context into account, we map $(B|A)[1]$ to $A \wedge (A \Rightarrow B)$, or equivalently, to AB. $(B|A)[0]$ is mapped to $\neg(A \Rightarrow B)$, or equivalently, to $A\overline{B}$, since $A \wedge \neg(A \Rightarrow B) = \neg(A \Rightarrow B)$. This yields the natural tranformation

$$\alpha_{\mathcal{C}/\mathcal{B}} : Sen_{\mathcal{C}} \Longrightarrow Sen_{\mathcal{B}} \qquad \text{with} \qquad \alpha_{\mathcal{C}/\mathcal{B},\Sigma}((B|A)[x]) = \begin{cases} AB & \text{if } x = 1 \\ A\overline{B} & \text{if } x = 0 \\ \perp & \text{otherwise} \end{cases}$$



Morphism	Sentence translation	Model translation
$Inst_{\mathcal{B}} \longrightarrow Inst_{\mathcal{P}}$	$A[x] \mapsto \begin{cases} A & \text{if } x = 1 \\ \overline{A} & \text{if } x = 0 \\ \bot & \text{otherwise} \end{cases}$	$I \mapsto P_I$
$Inst_{\mathcal{C}} \longrightarrow Inst_{\mathcal{P}}$	$A[x] \mapsto (A\|\top)[x]$	$P \mapsto P$
$Inst_{\mathcal{B}} \longrightarrow Inst_{\mathcal{C}}$	$(B\|A)[x] \mapsto \begin{cases} AB & \text{if } x = 1 \\ A\overline{B} & \text{if } x = 0 \\ \bot & \text{otherwise} \end{cases}$	$I \mapsto P_I$
$Inst_{\mathcal{B}} \longrightarrow Inst_{\mathcal{K}}$	$(B\|A) \mapsto AB$	$I \mapsto R_I$
$Inst_{\mathcal{C}} \longrightarrow Inst_{\mathcal{K}}$	$(B\|A) \mapsto (B\|A)[1]$	$P \mapsto R_P$

Fig. 1. Institution morphisms between $Inst_{\mathcal{B}}$, $Inst_{\mathcal{P}}$, $Inst_{\mathcal{C}}$, and $Inst_{\mathcal{K}}$

Proposition 11. $\langle\, id_{Sig_{\mathcal{B}}},\, \alpha_{\mathcal{C}/\mathcal{B}},\, \beta_{\mathcal{B}/\mathcal{C}} \,\rangle : Inst_{\mathcal{B}} \longrightarrow Inst_{\mathcal{C}}$ *is an institution morphism.*

Note that $\alpha_{\mathcal{C}/\mathcal{B},\Sigma}$ reflects the three-valued semantics of conditionals, identifying the verifying part AB and the falsifying part $A\overline{B}$ as most important components of conditional information (cf. [4, 3]).

Figure 1 summarizes our findings with respect to institution morphisms between the four institutions, where the relationships involving $Inst_{\mathcal{P}}$ are investigated in [1]. Using the intuitive standard translations, we have (essentially) exactly one institution morphism between any pair of the four institutions connected by arrows in Figure 1, but none going in the respective opposite direction. Moreover, the diagram is a commuting one; for instance, $\alpha_{\mathcal{K}/\mathcal{B}}$ is the (vertical) composition of the standard sentence translations $\alpha_{\mathcal{K}/\mathcal{C}}$ and $\alpha_{\mathcal{C}/\mathcal{B}}$, sending $(B|A)$ first to $(B|A)[1]$ and then to AB. Correspondingly, $\beta_{\mathcal{B}/\mathcal{K}}$ is the composition of the standard model translations $\beta_{\mathcal{B}/\mathcal{P}}$ and $\beta_{\mathcal{C}/\mathcal{K}}$, sending I first to P_I and then to R_I.

4 Conclusions and Further Work

In this paper, we used the general framework of institutions to formalize qualitative and probabilistic conditional logic as abstract logical systems. This allowed us to study the structural properties of both syntax and semantics of these logics, telling us, e.g., how conditionals are interpreted under change of notation. Moreover, in making use of the formal vehicle of institution morphisms, we investigated how qualitative and probabilistic conditionals and their respective models can be related to one another, and to the underlying two-valued propositional logic.

For qualitative as well as for probabilistic conditionals, the semantics we based our considerations on are quite standard, in order to make our results most widely applicable. However, there are lots of different semantics for conditionals, and it is an interesting question whether other semantics can yield different relationships between the involved logics. This is a topic of our ongoing research.

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