A logical study on qualitative default reasoning with probabilities

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Abstract. Only very special subclasses of probability distributions can be used for qualitative reasoning that meets basic logical demands. Snow's *atomic bound systems* (*big-stepped probabilities*) provide one positive example for such a subclass. This paper presents a thorough investigation of the formal logical relationships between qualitative and probabilistic default reasoning. We start with formalizing qualitative conditional logic, as well as both standard and big-stepped probabilistic logic as abstract logical systems, using the notion of *institutions*. The institution of big-stepped probabilities turns out to be a proper combination of the other two. Moreover, the framework of institutions offers the possibility to elaborate exactly the properties that make probability distributions suitable for qualitative reasoning.

1 Introduction

Default reasoning means walking on unstable logical grounds: Conclusions are drawn only defeasibly and may be revised when new information becomes evident. So the resulting logics are nonmonotonic [Mak94] and are closely related to belief revision theory [MG91]. On the other hand, they also make use of results and techniques from conditional logics [BDP97]. Indeed, a conditional relationship "If A then B", formally often denoted as $(B|A)$, is a most concise and complex piece of information. It may represent a default rule, a plausible relationship, a nonmonotonic inference, or a revision policy, depending on the context in which it is used [KI01]. So conditional logics [NC02], devised as logical systems to deal with if-then-statements in a non-classical way, have interesting connections to central concerns of Artificial Intelligence.

Roughly, two ways of conditional reasoning may be distinguished: qualitative approaches, aiming at following the traditions of symbolic logics, and quantitative approaches, making use of numerical information to handle conditionals. The oldest and most founded quantitative conditional theory is based on conditional probabilities. Whereas classical logics proved to be unable to represent conditionals properly, probability theory offered a convincing concept to deal with uncertain rules. In fact,

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one principal motivation for qualitative conditional reasoning was to handle conditionals in a non-classical way as easily and properly as probability theory does, but without probabilities. On the other hand, the rich expressiveness of probabilistics is sometimes perceived as a burden, so probabilists have attempted to use probabilities for a more qualitative kind of reasoning; the *comparative probabilities* of de Finetti [dF37,DP96] provide an early example for this. A more recent approach are the *atomic bound systems* of Snow [Sno94], more intuitively called *big-stepped probabilities*. A big-stepped probability distribution accepts a conditional $(B|A)$ iff its confirmation, $A \wedge B$, is more probable than its refutation, $A \wedge \neg B$; its specific probabilistic structure makes this compatible with basic standards of symbolic nonmonotonic reasoning.

This paper presents a rigorous logical investigation of qualitative reasoning with probabilities. Although qualitative and probabilistic conditional logics are substantially different, there is a framework broad enough to provide logical grounds for both of them. This is the framework of *institutions* which were introduced in [GB92] as a general framework for logical systems. Institutions are based on category theory and abstract from logical details, thus opening a unifying view on fundamental logical features. An institution formalizes the informal notion of a logical system, including syntax, semantics, and the relation of satisfaction between them. The latter poses the major requirement for an institution: that the satisfaction relation is consistent under the change of notation.

We will show in this paper, that not only qualitative and probabilistic conditional logics, but also the conditional logic of big-stepped probabilities can be formalized as institutions. Hereby, we build on work previously done [BKI02]. Then, by using the notion of institution morphisms [GR02], we study formal logical relationships between the institution of big-stepped probabilities, and the institution of qualitative conditional logic, being based on a semantics of total preorders. It will turn out, that there is exactly one such institution morphism that takes obvious similarities between both approaches properly into account. Furthermore, we raise a more general question that can be handled most adequately in the framework of institutions: Which probability distributions can be used to realize qualitative conditional reasoning? In this paper, we give an exact answer to this question, yielding a slight generalization of big-stepped probabilities.

The organization of this paper is as follows: In Section 2, we recall some basic definitions and facts from the theory of institutions. Section 3 gives a very brief overview on semantics for conditionals and shows how ordinal conditional logic and probabilistic conditional logic can be formalized as institutions. Section 4 then deals with qualitative probabilistic approaches; the institution of big-stepped probabilistic logic is set up as a combination of ordinal and probabilistic conditional logics. We investigate formal relationships between the institutions of big-stepped probabilistic logic and ordinal conditional logic, respectively, in Section 5. Section 6 summarizes the main results and points out further work.

2 The framework of institutions

Institutions will provide the general framework for this paper. After recalling some basic concepts of category theory and institutions, we present propositional logic as an institution. This will serve as a simple example, and provides us with useful notations.

2.1 Preliminaries: Basic Definitions and Notations

If C is a category, $|C|$ denotes the objects of C and $/C/$ its morphisms; for both objects $c \in |C|$ and morphisms $\varphi \in /C/$, we also write just $c \in C$ and $\varphi \in C$, respectively. C^{op} is the opposite category of C, with the direction of all morphisms reversed. The composition of two functors $F : C \to C'$ and $G : C' \to C''$ is denoted by $G \circ F$ (first apply F, then G). For functors $F, G : C \to C'$, a *natural transformation* η from F to G, denoted by $\eta : F \Longrightarrow G$, assigns to each object $c \in |C|$ a morphism $\eta_c : F(C) \to G(C) \in /C'/$ such that for every morphism $\varphi : c \to d \in /C/$ we have $\eta_d \circ F(\varphi) = G(\varphi) \circ \eta_c$. $\mathcal{SET}$ and $\mathcal{CAT}$ denote the categories of sets and of categories, respectively. (For more information about categories, see e.g. [HS73].) The central institution definition is the following (cf. Figure 1 that visualizes the relationships within an institution):

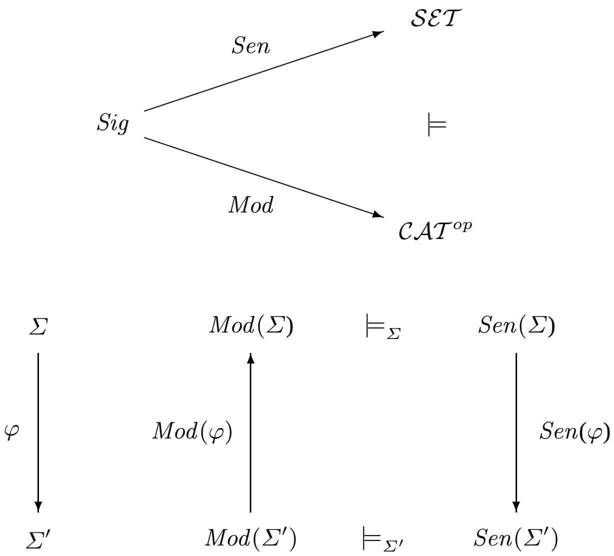

Fig. 1. Relationships within an institution $Inst = \langle\, Sig,\ Mod,\ Sen,\ \models\, \rangle$ [GB92]

Definition 1. [GB92] *An institution is a quadruple* $Inst = \langle\, Sig,\ Mod,\ Sen,\ \models\, \rangle$ *with a category* Sig *of signatures as objects, a functor* $Mod : Sig \to \mathcal{CAT}^{op}$ *yielding the category of* Σ*-models for each signature* Σ*, a functor* $Sen : Sig \to \mathcal{SET}$ *yielding* *the sentences over a signature, and a* $|\,Sig\,|$*-indexed relation* $\models_\Sigma \;\subseteq\; |\,Mod(\Sigma)\,| \times Sen(\Sigma)$ *such that for each signature morphism* $\varphi : \Sigma \to \Sigma' \in /Sig/$*, for each* $m' \in |\,Mod(\Sigma')\,|$*, and for each* $f \in Sen(\Sigma)$ *the following* satisfaction condition *holds:* $m' \models_{\Sigma'} Sen(\varphi)(f) \quad$ *iff* $\quad Mod(\varphi)(m') \models_\Sigma f$.

For sets F, G of Σ-sentences and a Σ-model m we write $m \models_\Sigma F$ iff $m \models_\Sigma f$ for all $f \in F$. The satisfaction relation is lifted to semantical entailment $\models_\Sigma$ between sentences by defining $F \models_\Sigma G$ iff for all Σ-models m with $m \models_\Sigma F$ we have $m \models_\Sigma G$. $F^\bullet = \{f \in Sen(\Sigma) \mid F \models_\Sigma f\}$ is called the *closure* of F, and F is *closed* if $F = F^\bullet$. The closure operator fulfils the *closure lemma* $\varphi(F^\bullet) \subseteq \varphi(F)^\bullet$ and various other nice properties like $\varphi(F^\bullet)^\bullet = \varphi(F)^\bullet$ or $(F^\bullet \cup G)^\bullet = (F \cup G)^\bullet$. A consequence of the closure lemma is that entailment is preserved under change of notation carried out by a signature morphism, i.e. $F \models_\Sigma G$ implies $\varphi(F) \models_{\varphi(\Sigma)} \varphi(G)$ (but not vice versa).

2.2 The Institution of Propositional Logic

In all circumstances, propositional logic seems to be the most basic logic. The components of its institution $Inst_\mathcal{B} = \langle\, Sig_\mathcal{B},\, Mod_\mathcal{B},\, Sen_\mathcal{B},\, \models_\mathcal{B} \,\rangle$ will be defined in the following.

Signatures: $Sig_\mathcal{B}$ is the category of propositional signatures. A propositional signature $\Sigma \in |\,Sig_\mathcal{B}\,|$ is a (finite) set of propositional variables, $\Sigma = \{a_1, a_2, \ldots\}$. A propositional signature morphism $\varphi : \Sigma \to \Sigma' \in /Sig_\mathcal{B}/$ is an injective function mapping propositional variables to propositional variables, allowing for the renaming of variables in a larger context.

Models: For each signature $\Sigma \in Sig_\mathcal{B}$, $Mod_\mathcal{B}(\Sigma)$ contains the set of all propositional interpretations for Σ, i.e. $|\,Mod_\mathcal{B}(\Sigma)\,| = \{I \mid I : \Sigma \to Bool\}$ where $Bool = \{\mathit{true}, \mathit{false}\}$. Due to its simple structure, the only morphisms in $Mod_\mathcal{B}(\Sigma)$ are the identity morphisms. For each signature morphism $\varphi : \Sigma \to \Sigma' \in Sig_\mathcal{B}$, we define the morphism (i.e. the functor in $\mathcal{CAT}^{op}$) $Mod_\mathcal{B}(\varphi) : Mod_\mathcal{B}(\Sigma') \to Mod_\mathcal{B}(\Sigma)$ by $(Mod_\mathcal{B}(\varphi)(I'))(a_i) := I'(\varphi(a_i))$ where $I' \in Mod_\mathcal{B}(\Sigma')$ and $a_i \in \Sigma$.

Sentences: For each signature $\Sigma \in Sig_\mathcal{B}$, the set $Sen_\mathcal{B}(\Sigma)$ contains the usual propositional formulas constructed from the propositional variables in Σ and the logical connectives $\wedge$ (and), $\vee$ (or), and $\neg$ (not). The symbols $\top$ and $\bot$ denote a tautology (like $a \vee \neg a$) and a contradiction (like $a \wedge \neg a$), respectively.

For each signature morphism $\varphi : \Sigma \to \Sigma' \in Sig_\mathcal{B}$, the function $Sen_\mathcal{B}(\varphi) : Sen_\mathcal{B}(\Sigma) \to Sen_\mathcal{B}(\Sigma')$ is defined by straightforward inductive extension on the structure of the formulas; e.g., $Sen_\mathcal{B}(\varphi)(a_i) = \varphi(a_i)$ and $Sen_\mathcal{B}(\varphi)(A \wedge B) = Sen_\mathcal{B}(\varphi)(A) \wedge Sen_\mathcal{B}(\varphi)(B)$. In the following, we will abbreviate $Sen_\mathcal{B}(\varphi)(A)$ by just writing $\varphi(A)$. In order to simplify notations, we will often replace conjunction by juxtaposition and indicate negation of a formula by barring it, i.e. $AB = A \wedge B$ and $\overline{A} = \neg A$. An *atomic formula* is a formula consisting of just a propositional variable, a *literal* is a positive or a negated atomic formula, and a *complete conjunction* is a conjunction of literals where all atomic formulas appear once, either in positive or in negated form. Ω_Σ denotes the set of all complete conjunctions over a signature Σ. Note that there is an obvious bijection between $|\, Mod_{\mathcal{B}}(\Sigma)\,|$ and Ω_{Σ}, so we will not distinguish between both sets.

Throughout the paper, two evident facts on formulas of the form $\varphi(\omega)$, where φ is a signature morphism, and ω is a complete conjunction, will often be used implicitly. First, $\varphi(\omega) \not\equiv \perp$, since φ is injective. Second, for two distinct complete conjunctions $\omega_1, \omega_2 \in \Omega_{\Sigma}$, the formulas $\varphi(\omega_1)$ and $\varphi(\omega_2)$ are exclusive, i.e. $\varphi(\omega_1)\varphi(\omega_2) \equiv \perp$.

Satisfaction relation: For any $\Sigma \in |\, Sig_{\mathcal{B}}\,|$, the satisfaction relation $\models_{\mathcal{B},\Sigma} \subseteq |\, Mod_{\mathcal{B}}(\Sigma)\,| \times Sen_{\mathcal{B}}(\Sigma)$ is defined as expected for propositional logic, e.g. $I \models_{\mathcal{B},\Sigma} a_i$ iff $I(a_i) = \textit{true}$ and $I \models_{\mathcal{B},\Sigma} A \wedge B$ iff $I \models_{\mathcal{B},\Sigma} A$ and $I \models_{\mathcal{B},\Sigma} B$ for $a_i \in \Sigma$ and $A, B \in Sen_{\mathcal{B}}(\Sigma)$. For ease of notation, we will write $\models$ instead of $\models_{\mathcal{B},\Sigma}$, if no confusion arises.

It is easy to check that $Inst_{\mathcal{B}} = \langle\, Sig_{\mathcal{B}},\, Mod_{\mathcal{B}},\, Sen_{\mathcal{B}},\, \models_{\mathcal{B}}\,\rangle$ is indeed an institution.

Example 1. Let $\Sigma = \{s, u\}$ and $\Sigma' = \{a, b, c\}$ be two propositional signatures with the atomic propositions *s – being a scholar*, *u – being single* and *a – being a student*, *b – being young*, *c – being unmarried*. Let I' be the Σ'-model with $I'(a) = \textit{true}$, $I'(b) = \textit{true}$, $I'(c) = \textit{false}$. Let $\varphi : \Sigma \to \Sigma' \in Sig_{\mathcal{B}}$ be the signature morphism with $\varphi(s) = a$, $\varphi(u) = c$. The functor $Mod_{\mathcal{B}}(\varphi)$ takes I' to the Σ-model $I := Mod_{\mathcal{B}}(\varphi)(I')$, yielding $I(s) = I'(a) = \textit{true}$ and $I(u) = I'(c) = \textit{false}$.

2.3 Institution Morphisms

An institution morphism Φ expresses a relation between two institutions $Inst$ und $Inst'$ such that the satisfaction condition of $Inst$ may be computed by the satisfaction condition of $Inst'$ if we translate it according to Φ. The translation is done by relating every $Inst$-signature Σ to an $Inst'$-signature Σ', each Σ'-sentence to a Σ-sentence, and each Σ-model to a Σ'-model.

Definition 2. [GB92] *Let*

$$Inst = \langle\, Sig,\, Mod,\, Sen,\, \models\,\rangle$$

and

$$Inst' = \langle\, Sig',\, Mod',\, Sen',\, \models'\,\rangle$$

be two institutions. An institution morphism *Φ from $Inst$ to $Inst'$ is a triple $\langle\, \phi,\, \alpha,\, \beta\,\rangle$ with a functor $\phi : Sig \to Sig'$, a natural transformation $\alpha : Sen' \circ \phi \Longrightarrow Sen$, and a natural transformation $\beta : Mod \Longrightarrow Mod' \circ \phi$ such that for each $\Sigma \in |\, Sig\,|$, for each $m \in |\, Mod(\Sigma)\,|$, and for each $f' \in Sen'(\phi(\Sigma))$ the following* satisfaction condition (for institution morphisms) *holds:*

$$m \models_{\Sigma} \alpha_{\Sigma}(f') \quad \textit{iff} \quad \beta_{\Sigma}(m) \models'_{\phi(\Sigma)} f'$$

Figure 2 illustrates the relationships within an institution morphism.

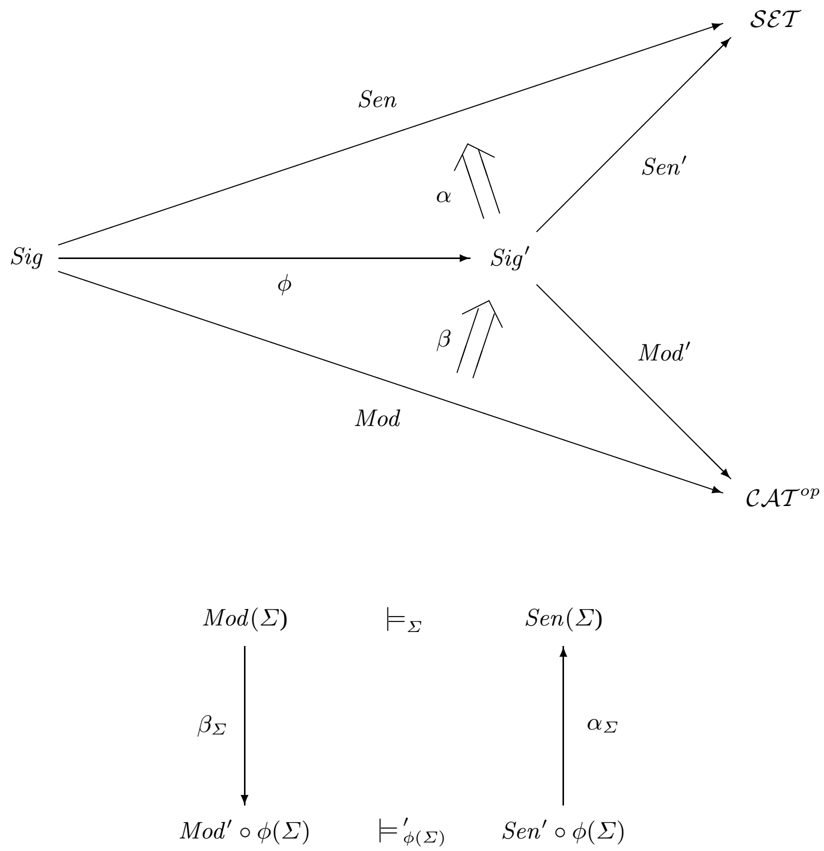

Fig. 2. Relationships within an institution morphism
$\langle\, \phi,\, \alpha,\, \beta \,\rangle : \langle\, Sig,\, Mod,\, Sen,\, \models \,\rangle \longrightarrow \langle\, Sig',\, Mod',\, Sen',\, \models' \,\rangle$

3 Semantics for conditionals

Conditionals $(B|A)$ (with propositional antecedent and consequent) can not be properly evaluated in classical logical environments. Instead, they substantially need the possibility to having their confirmation, AB, compared to their refutation, $A\overline{B}$, with respect to normality, plausibility, probability, and the like. This comparison can be done directly by using (pre)orders on formulas, or indirectly via appropriate numerical degrees of belief. In this section, we will present two well-known semantics for conditionals in the institution framework, namely, a qualitative semantics based on total preorders, and the usual probabilistic semantics making use of conditional probabilities. These results are based on work previously done [BKI02], but are briefly summarized here, since their notations and techniques will be used in the sequel.

3.1 The institution of qualitative conditional logic

Various types of models have been proposed to interpret conditionals adequately within a formal system (cf. e.g. [NC02]). Many of them are based on considering possible worlds which can be thought of as being represented by classical logical interpretations $|\,Mod_{\mathcal{B}}(\Sigma)\,|$, or complete conjunctions $\omega \in \Omega_\Sigma$ (as defined in Sec. 2.2), respectively. One of the most prominent approaches is the *system-of-spheres* model of Lewis [Lew73] which makes use of a notion of similarity between possible worlds. This idea of comparing worlds and evaluating conditionals with respect to the "nearest" or "best" worlds (which are somehow selected) is common to very many approaches in conditional logics.

The institution $Inst_{\mathcal{K}} = \langle\, Sig_{\mathcal{K}},\, Mod_{\mathcal{K}},\, Sen_{\mathcal{K}},\, \models_{\mathcal{K}}\,\rangle$, as specified in the following, provides a formal logical framework to carry out qualitative conditional reasoning.

Signatures: $Sig_{\mathcal{K}}$ is identical to the category of propositional signatures, i.e. $Sig_{\mathcal{K}} = Sig_{\mathcal{B}}$.

Models: In order to base our conditional logic on quite a general semantics, we take the models to be total preorders[1] over possible worlds, ordering them according to their *plausibility*, i.e.

$$|\,Mod_{\mathcal{K}}(\Sigma)\,| = \{R \mid R \text{ is a total preorder on } \Omega_\Sigma\}$$

By convention, the least worlds are the most plausible worlds. We will also use the infix notation $\omega_1 \preceq_R \omega_2$ instead of $(\omega_1, \omega_2) \in R$. As usual, we introduce the $\prec_R$-relation by saying that $\omega_1 \prec_R \omega_2$ iff $\omega_1 \preceq_R \omega_2$ and not $\omega_2 \preceq_R \omega_1$. Furthermore, $\omega_1 \approx_R \omega_2$ means that both $\omega_1 \preceq_R \omega_2$ and $\omega_2 \preceq_R \omega_1$ hold.

Each $R \in Mod_{\mathcal{K}}(\Sigma)$ induces a partitioning $\Omega_0, \Omega_1, \ldots$ of Ω, such that all worlds in the same partitioning subset are considered equally plausible ($\omega_1 \approx_R \omega_2$ for $\omega_1, \omega_2 \in \Omega_j$), and whenever $\omega_1 \in \Omega_i$ and $\omega_2 \in \Omega_k$ with $i < k$, then $\omega_1 \prec_R \omega_2$. Moreover, each $R \in Mod_{\mathcal{K}}(\Sigma)$ gives rise to a total preorder on $Sen_{\mathcal{B}}(\Sigma)$ by setting

$$A \preceq_R B \text{ iff } \exists\, \omega_0 \models A,\ \forall \omega \models B : \omega_0 \preceq_R \omega \tag{1}$$

So, A is considered to be at least as plausible as B (with respect to R) iff the most plausible worlds satisfying A are at least as plausible as any world satisfying B. Again, $A \prec_R B$ means both $A \preceq_R B$ and not $B \preceq_R A$. Note that $A \prec_R \bot$ for all $A \not\equiv \bot$. As before, we only consider the identity morphisms in $Mod_{\mathcal{K}}(\Sigma)$ for this paper.

For each signature morphism $\varphi : \Sigma \to \Sigma'$, we define a functor $Mod_{\mathcal{K}}(\varphi) : Mod_{\mathcal{K}}(\Sigma') \to Mod_{\mathcal{K}}(\Sigma)$ by mapping a (total) preorder R' over $Mod_{\mathcal{B}}(\Sigma')$ to a (total) preorder $Mod_{\mathcal{K}}(\varphi)(R')$ over $Mod_{\mathcal{B}}(\Sigma)$ in the following way:

$$\omega_1 \preceq_{Mod_{\mathcal{K}}(\varphi)(R')} \omega_2 \quad \text{iff} \quad \varphi(\omega_1) \preceq_{R'} \varphi(\omega_2) \tag{2}$$

Sentences: For each signature Σ, the set $Sen_{\mathcal{K}}(\Sigma)$ contains (*propositional*) *conditionals* of the form $(B|A)$ where $A, B \in Sen_{\mathcal{B}}(\Sigma)$ are propositional formulas from $Inst_{\mathcal{B}}$.

[1] A total preorder R is a reflexive and transitive relation such that for any two elements ω_1, ω_2, we have $(\omega_1, \omega_2) \in R$ or $(\omega_2, \omega_1) \in R$ (possibly both).

For $\varphi : \Sigma \to \Sigma'$, the extension $Sen_{\mathcal{K}}(\varphi)$ is defined as usual by $Sen_{\mathcal{K}}(\varphi)((B|A)) = (\varphi(B)|\varphi(A))$.

Satisfaction relation: The satisfaction relation $\models_{\mathcal{K},\Sigma} \subseteq |\,Mod_{\mathcal{K}}(\Sigma)\,| \times Sen_{\mathcal{K}}(\Sigma)$ is defined, for any $\Sigma \in |\,Sig_{\mathcal{K}}\,|$, by

$$R \models_{\mathcal{K},\Sigma} (B|A) \quad \text{iff} \quad AB \prec_R A\overline{B}$$

Therefore, a conditional $(B|A)$ is satisfied (or accepted) by the plausibility preorder R iff its confirmation AB is more plausible than its refutation $A\overline{B}$.

Example 2. We continue our student-example in this qualitative conditional environment, so let Σ, Σ', φ be as defined in Example 1. Let R' be the following total preorder on Ω':

$$\begin{aligned} R' : \quad & \overline{a}\overline{b}\overline{c} \prec_{R'} abc \approx_{R'} \overline{a}bc \\ & \prec_{R'} ab\overline{c} \approx_{R'} a\overline{b}c \approx_{R'} a\overline{b}\overline{c} \approx_{R'} \overline{a}b\overline{c} \approx_{R'} \overline{a}\overline{b}c \end{aligned}$$

For instance $R' \models_{\mathcal{K},\Sigma'} (c|a)$ – *students* are supposed to be *unmarried* since under R', ac is more plausible than $a\overline{c}$. Under $Mod_{\mathcal{K}}(\varphi)$, R' is mapped onto $R = Mod_{\mathcal{K}}(\varphi)(R')$ where R is the following total preorder on Ω:

$$R : \quad \overline{s}\,\overline{u} \prec_R \overline{s}u \approx_R su \prec_R s\overline{u}$$

As expected, the conditional $(u|s)$ that corresponds to $(c|a)$ in $Sen_{\mathcal{K}}(\Sigma')$ under φ, is satisfied by R – here, *scholars* are supposed to be *single*.

3.2 The institution of probabilistic conditional logic

A full probabilistic semantics for conditionals with precise probability values is provided by standard probability distributions P and conditional probabilities. In the language of institutions, the corresponding institution is given by $Inst_{\mathcal{C}} = \langle\, Sig_{\mathcal{C}},\, Mod_{\mathcal{C}},\, Sen_{\mathcal{C}},\, \models_{\mathcal{C}} \,\rangle$ with the following components: $Sig_{\mathcal{C}} = Sig_{\mathcal{B}}$ is the category of propositional signatures. The model functor $Mod_{\mathcal{C}}$ assigns to each signature Σ the category of probability distributions over Σ (with trivial morphisms). For each signature morphism $\varphi : \Sigma \to \Sigma'$, we define a functor $Mod_{\mathcal{C}}(\varphi) : Mod_{\mathcal{C}}(\Sigma') \to Mod_{\mathcal{C}}(\Sigma)$ by mapping each distribution P' over Σ' to a distribution $Mod_{\mathcal{C}}(\varphi)(P')$ over Σ. $Mod_{\mathcal{C}}(\varphi)(P')$ is defined by giving its value for all complete conjunctions ω over Σ:

$$(Mod_{\mathcal{C}}(\varphi)(P'))(\omega) \;:=\; P'(\varphi(\omega)) \tag{3}$$

The set $Sen_{\mathcal{C}}(\Sigma)$ of sentences consists of *probabilistic conditionals* of the form $(B|A)[x]$ with propositional formulas A, B and probabilities $x \in [0,1]$. For each signature morphism $\varphi : \Sigma \to \Sigma'$, the extension $Sen_{\mathcal{C}}(\varphi) : Sen_{\mathcal{C}}(\Sigma) \to Sen_{\mathcal{C}}(\Sigma')$ is defined by straightforward inductive extension on the structure of the formulas: $Sen_{\mathcal{C}}(\varphi)((B|A)[x]) = (\varphi(B)|\varphi(A))[x]$. Finally, the satisfaction relation $\models_{\mathcal{C},\Sigma} \subseteq |\,Mod_{\mathcal{C}}(\Sigma)\,| \times Sen_{\mathcal{C}}(\Sigma)$ is defined, for any $\Sigma \in |\,Sig_{\mathcal{C}}\,|$, by

$$P \models_{\mathcal{C},\Sigma} (B|A)[x] \quad \text{iff} \quad P(A) > 0 \text{ and } P(B|A) = \frac{P(AB)}{P(A)} = x$$

For more details and examples for $Inst_{\mathcal{C}}$, we refer to [BKI02].

4 Qualitative probabilities

Adams was the first to present a probabilistic framework for qualitative default reasoning. In his work [Ada75], he used an infinitesimal approach to define "reasonable (probabilistic) consequences". On these ideas, Pearl later based his ϵ-*semantics* [Pea89] which turned out to be the same as *preferential semantics* and can be characterized by the properties of *system P* [KLM90]. Therefore, the infinitesimal ϵ-semantics provides a probabilistic semantics for system P.

This seemed hardly possible to realize within a standard probabilistic framework. An obvious way to interpret a default rule "usually, if A then B", or "from A, defeasibly infer B" (written as $A \mid\!\sim B$) by a probability distribution P would be to postulate $P(AB) > P(A\overline{B})$ (which is equivalent to $P(B|A) > 0.5$). I.e. given A, the presence of B should be more probable than its absence. This interpretation, however, is not generally compatible with system P, it may conflict, for instance, with the *OR*-postulate of system P:

OR if $A \mid\!\sim C$ and $B \mid\!\sim C$ then $A \vee B \mid\!\sim C$

Indeed, it is easy to find counterexamples where $P(AC) > P(A\overline{C})$ and $P(BC) > P(B\overline{C})$, but $P((A \vee B)C) < P((A \vee B)\overline{C})$. So, in order to give reasonable probabilistic meanings to defaults, one has to focus on special subclasses of probability distributions.

Atomic bound systems, introduced by Snow in [Sno94], turned out to be such proper subclasses. Their distributions are also known as *big-stepped probabilities* (this more intuitive name was coined by Benferhat, Dubois & Prade, see [BDP99]).

Definition 3. *A* big-stepped probability distribution P over a signature Σ *is a probability distribution on* Ω_Σ *such that the following conditions are satisfied for all* $\omega, \omega_0, \omega_1, \omega_2 \in \Omega_\Sigma$*:*

$$P(\omega) > 0 \tag{4}$$

$$P(\omega_1) = P(\omega_2) \quad \textit{iff} \quad \omega_1 = \omega_2 \tag{5}$$

$$P(\omega_0) \;>\; \sum_{\omega : P(\omega_0) > P(\omega)} P(\omega) \tag{6}$$

The set of all big-stepped probability distributions over Σ *is denoted by* $\mathcal{P}_{BS}(\Sigma)$*.*

The last condition (6) explains the attribute "big-stepped": In a big-stepped probability distribution, the probability of each possible world is bigger than the sum of all probabilities of less probable worlds. Big-stepped probabilities actually provide a standard probabilistic semantics for system P, as was shown in [BDP99], by interpreting conditionals in the intended way:

$$P \models_{BS} (B|A) \quad \text{iff} \quad P(AB) > P(A\overline{B}) \tag{7}$$

for $P \in \mathcal{P}_{BS}$. The following two lemmata give a more detailed impression of properties of big-stepped probabilities, and of their resemblance to ordinal presentations of belief; their proofs are straightforward.

Lemma 1. *Let* $P \in \mathcal{P}_{BS}(\Sigma)$ *be a big-stepped probability distribution over a signature* Σ*, and let* A, B *be two propositional formulas. Then* $P(A) = P(B)$ *iff* $A \equiv B$*.*

Lemma 2. *Let $P \in \mathcal{P}_{BS}(\Sigma)$ be a big-stepped probability distribution over a signature Σ, and let A, B be two non-contradictory, exclusive propositional formulas. Then*

$$P(A) > P(B) \textit{ iff } \exists \omega_0 \models A, \forall \omega \models B : P(\omega_0) > P(\omega) \tag{8}$$

These lemmata will prove useful in the sequel. We are now ready to set up the institution $Inst_{\mathcal{S}}$ of big-stepped probabilities by deriving its components both from $Inst_{\mathcal{C}}$ and $Inst_{\mathcal{K}}$:

- $Sig_{\mathcal{S}} = Sig_{\mathcal{B}}$, i.e. the signatures are propositional signatures.
- $Mod_{\mathcal{S}}(\Sigma)$ is the full subcategory obtained from $Mod_{\mathcal{C}}(\Sigma)$ by restriction to the big-stepped probability distributions $\mathcal{P}_{BS}(\Sigma)$; for each signature morphism $\varphi : \Sigma \to \Sigma'$, the functor $Mod_{\mathcal{S}}(\varphi)$ is the restriction and corestriction of $Mod_{\mathcal{C}}(\varphi)$ to $Mod_{\mathcal{S}}(\Sigma')$ and $Mod_{\mathcal{S}}(\Sigma)$.
- $Sen_{\mathcal{S}} = Sen_{\mathcal{K}}$, i.e. the sentences are conditionals as in $Inst_{\mathcal{K}}$ with corresponding sentence translation.
- In correspondence to (7), the satisfaction relation is defined by: $P \models_{\mathcal{S},\Sigma} (B|A)$ iff $P(AB) > P(A\overline{B})$.

Proposition 1. $Inst_{\mathcal{S}} = \langle\, Sig_{\mathcal{S}},\, Mod_{\mathcal{S}},\, Sen_{\mathcal{S}},\, \models_{\mathcal{S}} \rangle$ *is an institution.*

Proof. With the help of Lemmata 1 and 2, and the injectivity of φ, it is straightforward to check that $Mod_{\mathcal{S}}(\varphi)$ is well-defined, i.e. mapping big-stepped probability distributions over Σ' to big-stepped probability distributions over Σ. (Note that the presupposition that signature morphisms are injective is crucial for this; otherwise, $\varphi(\omega)$ may be contradictory for possible worlds ω, in which case zero probabilities would arise.) What remains to be shown is that the satisfaction condition from Definition 1 is satisfied, which in this case requires

$$P' \models_{\mathcal{S},\Sigma'} Sen_{\mathcal{S}}(\varphi)(B|A) \;\text{ iff }\; Mod_{\mathcal{S}}(\varphi)(P') \models_{\mathcal{S},\Sigma} (B|A)$$

to hold. But this is trivial, since $Sen_{\mathcal{S}}(\varphi)(B|A) = (\varphi(B)|\varphi(A))$ and $Mod_{\mathcal{S}}(\varphi)(P')(B|A) = P'(\varphi(B)|\varphi(A))$.

5 Relating ordinal and probabilistic qualitative logics

Now that we have formalized both ordinal and qualitative probabilistic logics for conditionals as institutions, we can use institution morphisms (cf. Section 2.3) to study formal logical relationships between them. In fact, the resemblance between $Inst_{\mathcal{K}}$ and $Inst_{\mathcal{S}}$ is quite close: Both have identical syntax, provided by propositional signatures $Sig_{\mathcal{K}} = Sig_{\mathcal{S}} = Sig_{\mathcal{B}}$ and conditional sentences $Sen_{\mathcal{K}} = Sen_{\mathcal{S}}$, but differ with respect to the semantics defined by the model functors $Mod_{\mathcal{K}}$ and $Mod_{\mathcal{S}}$, and the satisfaction relations $\models_{\mathcal{K}}$ and $\models_{\mathcal{S}}$, respectively. Therefore, we will focus on institution morphisms $\langle\, \phi,\, \alpha,\, \beta \,\rangle$ that take this resemblance into account by having $\phi = id_{Sig_{\mathcal{B}}}$ and $\alpha = id_{Sen_{\mathcal{K}}}$. So, only appropriate natural transformations β relating the different semantics have to be studied. In view of the similarity of the crucial relations (1) and (8), this might be expected to be trivial. However, a first negative result is stated in the following proposition:

Proposition 2. *There is no institution morphism* $\langle\, id_{Sig_{\mathcal{B}}},\, id_{Sen_{\mathcal{K}}},\, \beta\,\rangle \;:\; Inst_{\mathcal{K}} \;\rightarrow\; Inst_{\mathcal{S}}$.

Proof. Assume there is such an institution morphism $\langle\, id_{Sig_{\mathcal{B}}},\, id_{Sen_{\mathcal{K}}},\, \beta\,\rangle : Inst_{\mathcal{K}} \rightarrow Inst_{\mathcal{S}}$. Then for each $R \in Mod_{\mathcal{K}}(\Sigma)$, $P_R := \beta_{\Sigma}(R) \in Mod_{\mathcal{S}}(\Sigma)$ is a big-stepped probability distribution. The satisfaction condition (for morphisms) requires $R \models_{\mathcal{K},\Sigma} (B|A)$ iff $P_R \models_{\mathcal{S},\Sigma} (B|A)$, i.e. $AB \prec_R A\overline{B}$ iff $P_R(AB) > P_R(A\overline{B})$, which implies $\omega_1 \prec_R \omega_2$ iff $P_R(\omega_1) > P_R(\omega_2)$ for $\omega_1, \omega_2 \in \Omega_{\Sigma}$. Now, it is obvious that we can choose an R such that $\omega_1 \approx_R \omega_2$ and $\omega_1 \neq \omega_2$, yielding $P(\omega_1) = P(\omega_2)$, which is impossible in $Inst_{\mathcal{S}}$.

Thus, there is no way of mapping ordinal models from $Inst_{\mathcal{K}}$ to big-stepped probabilistic models in $Inst_{\mathcal{S}}$ within such an institution morphism.

Fortunately, institution morphisms in the other direction turn out to be feasible. An obvious way to define a natural transformation $\beta : Mod_{\mathcal{S}} \Longrightarrow Mod_{\mathcal{K}}$ is to associate to each big-stepped probability $P \in Mod_{\mathcal{S}}(\Sigma)$ a total preorder $R_P \in Mod_{\mathcal{K}}(\Sigma)$ via

$$\omega_1 \preceq_{R_P} \omega_2 \text{ iff } P(\omega_1) \geqslant P(\omega_2) \tag{9}$$

The next proposition shows this to be (uniquely) successful.

Proposition 3. $\langle\, id_{Sig_{\mathcal{B}}},\, id_{Sen_{\mathcal{K}}},\, \beta_{\mathcal{S}/\mathcal{K}}\,\rangle : Inst_{\mathcal{S}} \rightarrow Inst_{\mathcal{K}}$ *with* $\beta_{\mathcal{S}/\mathcal{K},\Sigma}(P) := R_P$ *as defined in (9) for each* $\Sigma \in Sig_{\mathcal{B}}$ *and* $P \in Mod_{\mathcal{S}}(\Sigma)$ *is the only institution morphism from* $Inst_{\mathcal{S}}$ *to* $Inst_{\mathcal{K}}$.

It is straightforward to check that $\langle\, id_{Sig_{\mathcal{B}}},\, id_{Sen_{\mathcal{K}}},\, \beta_{\mathcal{S}/\mathcal{K}}\,\rangle$ is indeed an institution morphism, and that the satisfaction condition leaves no other possibility.

So, also on formal logical grounds, the big-stepped probability distributions prove to be adequate to implement qualitative conditional reasoning. The institution morphism from $Inst_{\mathcal{S}}$ to $Inst_{\mathcal{K}}$ makes probabilistic reasoning with big-stepped probabilities fully compatible to qualitative reasoning based on total preorders.

With the instruments of institutions and institution morphisms at hand, we may also consider a more general question concerning qualitative probabilities: Which are the properties that make probability distributions adequate for qualitative reasoning? More exactly, which subclasses $Mod_{\mathcal{Q}}(\Sigma)$ of $Mod_{\mathcal{C}}(\Sigma)$ are apt to base an institution $Inst_{\mathcal{Q}}$ on them, similarly to $Inst_{\mathcal{S}}$ (in particular, with a qualitative satisfaction relation corresponding to (7)) and also allowing a morphism from $Inst_{\mathcal{Q}}$ to $Inst_{\mathcal{K}}$? The following proposition gives a comprehensive answer to this question.

Proposition 4. *Let* $Inst_{\mathcal{Q}} = \langle\, Sig_{\mathcal{Q}},\, Mod_{\mathcal{Q}},\, Sen_{\mathcal{Q}},\, \models_{\mathcal{Q}}\,\rangle$ *be an institution with* $Sig_{\mathcal{Q}} = Sig_{\mathcal{B}}$, $|\,Mod_{\mathcal{Q}}(\Sigma)\,| \subseteq |\,Mod_{\mathcal{C}}(\Sigma)\,|$, $Mod_{\mathcal{Q}}(\varphi)$ *the restriction and corestriction of* $Mod_{\mathcal{C}}(\varphi)$, $Sen_{\mathcal{Q}} = Sen_{\mathcal{K}}$, *and* $\models_{\mathcal{Q}}$ *in accordance with (7).*

If there is an institution morphism

$$\langle\, id_{Sig_{\mathcal{B}}},\, id_{Sen_{\mathcal{K}}},\, \beta\,\rangle : Inst_{\mathcal{Q}} \rightarrow Inst_{\mathcal{K}}$$

then the probability distributions P *from* $|\,Mod_{\mathcal{Q}}(\Sigma)\,|$ *have to satisfy the properties (4) and (6), and for all* $\omega_1, \omega_2 \in \Omega$, *it holds that*

$$\begin{aligned} &P(\omega_1) = P(\omega_2) \textit{ iff } \omega_1 = \omega_2, \textit{ or for all} \\ &\omega \in \Omega_\Sigma \backslash \{\omega_1, \omega_2\}, P(\omega) > P(\omega_1) = P(\omega_2) \end{aligned} \tag{10}$$

Conversely, the subclass of $|\,Mod_{\mathcal{C}}(\Sigma)\,|$ *consisting of all probability distributions with properties (4), (6), and (10) gives rise to an institution in the specified way.*

Proof. The presupposed existence of the institution morphism from $Inst_{\mathcal{Q}}$ to $Inst_{\mathcal{K}}$ ensures that with each probability distribution $P \in Mod_{\mathcal{Q}}(\Sigma)$, a total preorder $\preceq_P := \beta_\Sigma(P)$ on Ω_Σ is associated such that $P \models_{\mathcal{Q},\Sigma} (B|A)$ iff $\beta_\Sigma(P) \models_{\mathcal{K},\Sigma} (B|A)$, i.e.

$$P(AB) > P(A\overline{B}) \quad \text{iff} \quad AB \prec_P A\overline{B} \tag{11}$$

The condition (11) has important consequences. First, considering the conditional $(\omega|\omega)$ yields $P(\omega) > 0$, since $\omega \prec_P \perp$ for all $\omega \in \Omega$. Furthermore, for $\omega_1, \omega_2 \in \Omega$, we obtain $P(\omega_1) > P(\omega_2)$ iff $\omega_1 \prec_P \omega_2$ by setting $A = \omega_1 \vee \omega_2, B = \omega_1$. In order to recover the crucial property (6), we consider $\omega_0 \in \Omega$ and set

$$A := \omega_0 \vee \bigvee \{\omega \mid P(\omega) < P(\omega_0)\}, \;\; B := \omega_0.$$

Then $AB = \omega_0$, $A\overline{B} = \{\omega \mid P(\omega) < P(\omega_0)\}$. Since for all $\omega \in A\overline{B}$, we have $P(\omega) < P(\omega_0)$, which implies $\omega_0 \prec_P \omega$, it holds that $AB \prec_P A\overline{B}$. Due to (11), we obtain

$$P(\omega_0) > \sum_{P(\omega_0) > P(\omega)} P(\omega)$$

which is (6). The property (10) is shown similarly, by considering suitable conditionals. Finally, it is straightforward to prove that the class of probability distributions so defined is apt to provide the basis for an institution $Inst_{\mathcal{Q}}$, and that the natural transformation β with $\beta_\Sigma(P) := R_P$ as in (9) again defines an institution morphism in the required way.

The postulated institutional compatibility of probabilities with qualitative default reasoning produces a slightly generalized version of big-stepped probability distributions: Instead of assuming that all probabilities are different (and hence linearly ordered), the probabilities of two possible worlds may now be identical, but this probability has to be the least one. This is the utmost concession that can be made; otherwise, the compatibility of the additive probabilistic structure with qualitative reasoning gets lost.

6 Summary and further work

This paper presented a thorough logical formalizations of different logics for conditionals. The concept of institutions, introduced by Goguen and Burstall [GB92], made it possible not only to study qualitative and probabilistic conditional logic within one framework, but also to combine them in order to specify the components of a qualitative probabilistic logic based on big-stepped probabilities. Moreover, the formal relationship to qualitative conditional logic via institution morphisms allows a precise characterization of those probability distributions which can be used for qualitative probabilistic reasoning.

As representative for qualitative conditional logics, we used a logic whose semantics is based on total preorders on possible worlds. In fact, there are many other well-known semantics for conditionals, for instance, the ordinal conditional functions of Spohn [Spo88], or possibilistic semantics (cf. e.g. [BDP99]). It is also possible to consider these logics as institutions, and the study of relationships between these different qualitative conditional logics via institution morphisms shows them to be quite similar. A thorough investigation of these aspects is subject of our current work.

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