Empirical evidence on the topological properties of structural paths and some notes on its theoretical explanation

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Abstract. The mathematical literature has developed a large pool of topological concepts and theorems for dynamic systems analysis. The aim of our paper is to make a first step towards the application of these concepts and theorems in the analysis of (long-run) structural change (in the three-sector framework). Our approach focuses on two of the most basic topological notions, namely intersection and self-intersection of trajectories on a two-dimensional domain. We discuss the mathematical foundations of the application of these concepts in structural change analysis, use them for analyzing empirical data, and elaborate new stylized facts stating that different countries' structural change trajectories are (non-self-)intersecting. Finally, we discuss briefly the theoretical explanations of (non-self-)intersection and a wide range of new research topics relating to (a) the topological classification and comparison of models and evidence and (b) the application of (further) topological concepts in standard branches of growth and development theory.

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1. Introduction

Structural change (and, in particular, long-run labor reallocation) in the three-sector framework (referring to the agricultural, manufacturing, and services sector) is a traditional topic of growth and development theory and has been analyzed in numerous models and empirical studies over the last centuries.[1] While the standard structural change literature relies on the mathematical branches of analysis and algebra for modeling structural change and describing the relevant empirical evidence, we suggest a *topological approach* for studying structural change. This seems to be a natural extension of the existing methods of structural change analysis, since a great part of the mathematical literature on dynamic systems (and its applications in physics and engineering) has reoriented towards topological methods over the last century creating a large pool of topological concepts and theorems that are potentially applicable in structural change modeling. Our paper aims to be a first step towards the application of topology in structural change analysis, demonstrating the applicability of basic topological concepts in empirics and theory of structural change and laying the foundations for the application of more sophisticated topological methods in this field (cf. Stijepic (2015c)). Moreover, even the relatively simple topological concepts and evidence discussed in our paper can be used for structural change modeling and prediction as demonstrated by Stijepic (2015, 2017c,d).

The first part of our paper deals with the conceptual and mathematical aspects of the topological approach. As discussed there, structural change (in a country) can be described by a trajectory on the standard two-dimensional simplex, where the trajectories (of different countries) can be characterized by the topological notions of self-intersection and intersection. Thus, empirical evidence and (existing) theoretical models can be classified and compared to each other by using these notions.

In the second part of our paper, we analyze the data on the long-run labor allocation dynamics in the OECD countries and formulate two new stylized facts stating that (a) the long-run labor allocation trajectories intersect and (b) self-intersection seems to be a short-run phenomenon and, thus, non-self-intersection is characteristic for the long run.

[1] For an overview of the structural change literature, see, e.g., Schettkat and Yocarini (2006), Krüger (2008), Silva and Teixeira (2008), Stijepic (2011, Chapter IV), Herrendorf et al. (2014), and van Neuss (2018). Recent papers modeling structural change in the three-sector framework are, e.g., Kongsamut et al. (2001), Ngai and Pissarides (2007), Foellmi and Zweimüller (2008), Uy et al. (2013), and Stijepic (2015, 2017d).

Since we are not aware of any literature that discusses or tries to theoretically explain the stylized facts derived in the second part of our paper,[2] we devote the third part of our paper to this topic. In other words, the third part deals with the comparison of theoretical models with empirical evidence. While the empirically observable intersections (of trajectories representing different countries) are not surprising from the theoretical point of view (if we assume that model parameters differ across countries; cf. Section 4.1), the long-run non-self-intersection seems to be an interesting theoretical puzzle. Therefore, we discuss briefly the theoretical and intuitive/economic explanations of non-self-intersection. In part, we discuss these aspects by relying on topological concepts (in particular, homeomorphisms).

Finally, we show that many standard topics of development and growth theory (ranging from savings rate dynamics and functional income distribution to wealth distribution and consumption structure dynamics) can be studied by applying our topological approach, indicating a great potential for further research in this field. Overall, our approach generates new evidence, new theoretical arguments, and numerous topics for further research (which are summarized in Section 5).

The rest of the paper is set up as follows. Section 2 deals with the conceptual and mathematical foundations of the topological approach. In Section 3, we present the evidence on labor reallocation focusing on OECD countries and the data provided by the World Bank and Maddison (1995) and formulate the stylized facts regarding the topological properties of labor allocation trajectories. Section 4 is devoted to the development of a theoretical intuitive/economic explanation of the observed stylized facts. A summary of our findings and a discussion of the topics for further research are provided in Section 5.

2. Geometrical interpretation of structural change and topological characterization of (families of) trajectories

In this section, we discuss the geometrical and topological concepts that can be used to describe and characterize structural change models and the empirical evidence on structural change. We start with a mathematical definition of structural change in Section 2.1. Then, we discuss (a) the geometrical representation of structural change (models) by simplexes and (families of) trajectories (cf. Section 2.2) and (b) some topological concepts that can be used to characterize (families of) trajectories and, thus, structural dynamics (cf. Section 2.3).

[2] Stijepic (2015) suggests a meta-model of non-self-intersecting trajectories and studies the transitional dynamics in this model. In contrast to Stijepic (2015), we focus on the empirical evidence and the theoretical explanation of non-self-intersection. Moreover, in contrast to Stijepic (2015) we discuss (non-)intersection.

While there are different mathematical notational conventions, we choose the following notation for reasons of simplicity: small letters (e.g., x), bold small letters (e.g., $\mathbf{x}$), and capital letters (e.g., X) denote scalars, vectors/points, and sets, respectively. A dot indicates a derivative with respect to time (e.g., $\dot{x}$ is the derivative of x with respect to time).

2.1 A mathematical definition of structure and structural change

We start with straight forward definitions of structure and structural change as used by Stijepic (2015, 2017d).

Definition 1. *Let $y(t)$ denote the aggregate employment at time t. Moreover, let $y_1(t)$, $y_2(t)$, and $y_3(t)$ stand for the employment in the agricultural, manufacturing, and services sector at time t, respectively, where $t \in D \subseteq R$ and R is the set of real numbers. Then, $x_i(t) := y_i(t)/y(t)$ represents the employment share of sector i for all $t \in D$ and for all $i \in \{1,2,3\}$. The 'structure' (of employment) at time $t \in D$ is represented by the vector $\mathbf{x}(t) := (x_1(t), x_2(t), x_3(t)) \in R^3$, where $\mathbf{x}(t)$ satisfies the following conditions:*

(1a) $\forall t \in D \ \forall i \in \{1,2,3\} \ 0 \leq x_i(t) \leq 1$

(1b) $\forall t \in D \ x_1(t) + x_2(t) + x_3(t) = 1$

Thus, Definition 1 states that the employment structure is simply a vector in 3-dimensional real space that satisfies the conditions (1). Standard models of structural change (cf. Footnote 1) satisfy conditions (1), in general.

Definition 2. *Structural change (over the period $[a,b]$) refers to the long-run dynamics of $\mathbf{x}(t)$ (over the period $[a,b]$; cf. Definition 1).*

Simply speaking, Definition 2 states that structural change takes place if $\mathbf{x}(t)$ is not constant in the long-run.

2.2 Geometrical interpretation of structure and structural change: simplexes and families of trajectories

In this section, we recapitulate some geometrical concepts for analyzing structural change (cf. Stijepic (2015)).

The set of all points $\mathbf{x}$ (in 3-dimensional real space) that satisfy Definition 1 is:

(2) $\{\mathbf{x} \equiv (x_1, x_2, x_3) \in R^3: x_1 + x_2 + x_3 = 1 \wedge \forall i \in \{1,2,3\}\ 0 \leq x_i \leq 1\} =: S$

It is well known that (2) is the definition of a standard 2-simplex (S), which is a triangle in the Cartesian coordinate system (x_1, x_2, x_3). The coordinates of its vertices are (cf. Figure 1):

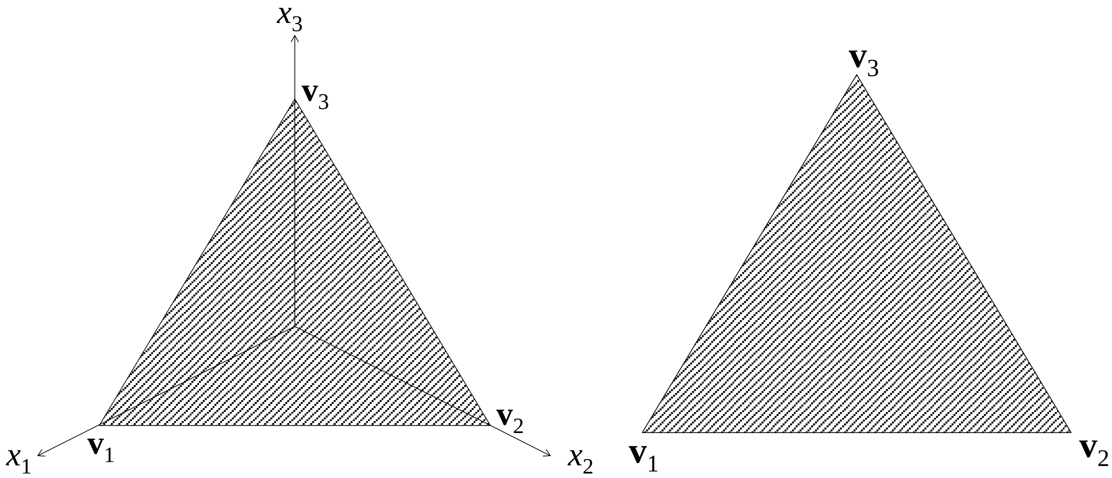
(3a) $(1, 0, 0) =: \mathbf{v}_1$

(3b) $(0, 1, 0) =: \mathbf{v}_2$

(3c) $(0, 0, 1) =: \mathbf{v}_3$

Henceforth, we omit the coordinate axes when depicting S, as illustrated by the right-hand panel of Figure 1.

Figure 1. *The 2-simplex in the Cartesian coordinate system* (x_1, x_2, x_3) *with and without coordinate axes.*



Definition 1 and (2) imply the following geometrical interpretation of the term structure: the employment structure (cf. Definition 1) can be represented by a point on the standard 2-simplex. This 2-simplex contains all the points that satisfy Definition 1. Two different points on the simplex represent two different structures. Thus, if, e.g., $\mathbf{x}(1) \neq \mathbf{x}(2)$ (cf. Definition 1), where $\mathbf{x}(1),\mathbf{x}(2) \in S$, then the structure at $t = 2$ is not the same as the structure at $t = 1$.

We turn now to a discussion of the representation of structural change via functions and trajectories on the standard simplex. Let us assume the following function:

(4a) $\phi: D \times P \times I \rightarrow S$

(4b) $\phi: (t,\mathbf{p},\mathbf{x}_0) \longmapsto \mathbf{x} \equiv (x_1, x_2, x_3)$

(4c) $\mathbf{x}_0 \in I \subseteq S$

where $\mathbf{p}$ is a parameter vector taking values in the set P and $\mathbf{x}_0$ is an index representing the initial condition of the system taking values in the set I. (4) states that the function $\phi(t,\mathbf{p},\mathbf{x}_0)$

maps time (t), the parameter vector ($\mathbf{p}$), and the initial condition vector ($\mathbf{x}_0$) to the 2-simplex. In particular, for a given initial condition $\mathbf{x}_0$ and a given parameter vector $\mathbf{p}\in P$, the function $\phi(t,\mathbf{p},\mathbf{x}_0)$ assigns to each time point $t\in D$ a point on the 2-simplex S, which is located in the coordinate system (x_1, x_2, x_3).

Standard structural change models (e.g., the models listed in Footnote 1) generate functions of the type (4) (see Appendix B for an example). Thus, (4) can be regarded as a structural change meta-model (covering different structural change models known from the literature). Since the function (4) assigns a structure to each point in time of the domain D (cf. (2), (4a), and Definition 1), we can derive all the information about structural change (cf. Definition 2) from this function. In particular, by studying $\phi(t,\mathbf{p},\mathbf{x}_0)$ we can derive how the structure changes over time for a given initial condition $\mathbf{x}_0$ and a given setting of the model parameters $\mathbf{p}$. Therefore, we focus on the analysis of this function henceforth.

To study the properties of the structural function $\phi(t,\mathbf{p},\mathbf{x}_0)$ geometrically, we use the concept of (the image of a) trajectory ($T(\mathbf{p},\mathbf{x}_0)$), which we define as follows (cf. Definition 1):

(5) $\forall\mathbf{x}_0\in I\ \forall\mathbf{p}\in P\ \ T(\mathbf{p},\mathbf{x}_0) := \{\phi(t,\mathbf{p},\mathbf{x}_0)\in S\colon t\in D\}$

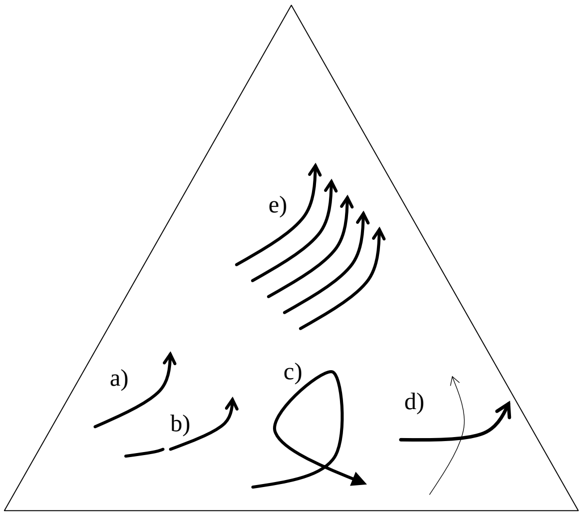
In fact, $T(\mathbf{p},\mathbf{x}_0)$ is simply the set of states (or: structures) that the economy experiences (or: goes through) over the time period D for the given initial condition $\mathbf{x}_0$ and the given parameter setting $\mathbf{p}$. Geometrically speaking, the economy moves along $T(\mathbf{p},\mathbf{x}_0)$ over the time period D if the initial condition is $\mathbf{x}_0$ and the parameter setting is $\mathbf{p}$. Note that (5) implies that the structural trajectory $T(\mathbf{p},\mathbf{x}_0)$ is always located on the standard simplex S. Thus, we can say that S is the *domain of the structural trajectory*.

Figure 2a depicts an example of a trajectory given by (4) and (5), where we assume that $\phi(t,\mathbf{p},\mathbf{x}_0)$ is continuous in t for a given initial condition $\mathbf{x}_0$ and a given parameter setting $\mathbf{p}$. Note that the arrows in Figure 2 indicate the direction of the movement along the trajectories.

(4) and (5) generate families of trajectories. For example, if we fix the parameter vector $\mathbf{p}$, (4) and (5) generate a family (I) of trajectories, where each trajectory belonging to the family I corresponds to one initial state $\mathbf{x}_0$ from the set I. Analogously, if we fix the initial vector $\mathbf{x}_0$, (4) and (5) generate a family (P) of trajectories, where each trajectory corresponds to a different parameter vector $\mathbf{p}$. Figures 2d and 2e depict families of trajectories, where we assume that $\phi(t,\mathbf{p},\mathbf{x}_0)$ is continuous in t.

Overall, the mathematical concepts elaborated in this section allow us to interpret a structural change model as a family of (parameter dependent) trajectories on the standard simplex.

Figure 2. *Examples of (families of) trajectories on S.*



2.3 Topological characterization of trajectory families: continuity and (self-)intersection

Trajectories can be characterized by using the topological concepts of continuity, self-intersection, and in the case of a family of trajectories, (mutual) intersection.

The intuitive/geometrical notion of a *continuous* trajectory is more or less obvious: it is a curve without interruptions (see, e.g., Figure 2a). In contrast, Figure 2b depicts an example of a non-continuous trajectory. The following definition of a continuous trajectory is obvious.

Definition 3. *The trajectory* $T(\mathbf{p},\mathbf{x}_0)$ *(cf. (5)) is continuous on S (for a given initial condition* $\mathbf{x}_0$ *and a given parameter setting* $\mathbf{p}$*) if the corresponding function* $\phi(t,\mathbf{p},\mathbf{x}_0)$ *(cf. (4)) is continuous (in t) on the interval D (for the initial condition* $\mathbf{x}_0$ *and the parameter setting* $\mathbf{p}$*). The trajectory family I (cf. (5)) is continuous on S (for the parameter setting* $\mathbf{p}$*) if for all* $\mathbf{x}_0 \in I$*,* $T(\mathbf{p},\mathbf{x}_0)$ *is continuous on S (for the parameter setting* $\mathbf{p}$*).*

The geometrical/intuitive meaning of the self-intersection of a (continuous) trajectory is more or less obvious: the trajectory in Figure 2a does not intersect itself, whereas the trajectory in Figure 2c intersects itself. We apply the following formal definition of non-self-intersection (cf. Stijepic (2015), p.82).

Definition 4. *The (continuous and non-closed) trajectory* $T(\mathbf{p},\mathbf{x}_0)$ *(cf. (5)) is non-self-intersecting (for a given initial condition* $\mathbf{x}_0$ *and a given parameter setting* $\mathbf{p}$*) if* $\nexists(t_1, t_2, t_3) \in D^3$: $t_1 < t_2 < t_3 \wedge \phi(t_1,\mathbf{p},\mathbf{x}_0) = \phi(t_3,\mathbf{p},\mathbf{x}_0) \neq \phi(t_2,\mathbf{p},\mathbf{x}_0) \wedge \mathbf{p} \in P \wedge \mathbf{x}_0 \in I$*.*

Note that per Definition 4, a self-intersection requires that the economy leaves the point $\phi(t_1, \mathbf{p},\mathbf{x}_0)$ at least for some instant of time (t_2) before it returns to it (at t_3). Thus, according to our definition, a self-intersection does not occur if the economy reaches some point on S (in finite time) and stays there forever. A second possibility to define a non-self-intersecting trajectory is a topological one: a non-self-intersecting trajectory is homeomorphic to the real line (cf. Section 4.2.1). Finally, we define a non-intersecting family of trajectories, as follows.

Definition 5. *The (continuous) trajectory family I (cf. (5)) is non-intersecting (for a given parameter setting* $\mathbf{p}$*) if* $\nexists(\mathbf{x}_0, \underline{\mathbf{x}}_0)\in I^2\text{: } \mathbf{x}_0 \neq \underline{\mathbf{x}}_0 \wedge T(\mathbf{p},\mathbf{x}_0)\cap T(\mathbf{p},\underline{\mathbf{x}}_0) \neq \emptyset \wedge \mathbf{p}\in P$.

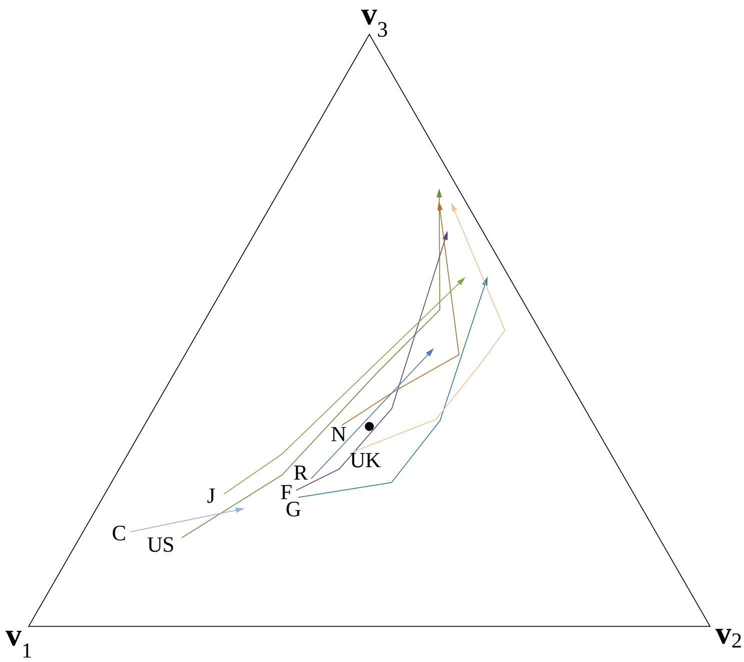
That is, if we choose two different trajectories ($\mathbf{x}_0 \neq \underline{\mathbf{x}}_0$) from the family I, they must not have a point of intersection (i.e., they must not occupy a common point on S) for a given parameter setting $\mathbf{p}$. Figure 2d depicts an intersecting family of trajectories (for a given $\mathbf{p}$), whereas Figure 2e depicts a non-intersecting family of trajectories (for a given $\mathbf{p}$).

3. Evidence on the topological properties of structural change trajectories

In accordance with (5), we construct the labor allocation trajectory of each country in our sample as follows (cf. Stijepic (2017e)). Assume that we have data on labor allocation ($\mathbf{x}(t)$) across agriculture, manufacturing, and services in county A for the time points t_0, t_1,…t_m. That is, we have the data points $\mathbf{x}(t_0)$, $\mathbf{x}(t_1)$,…$\mathbf{x}(t_m)$ associated with country A. We construct the labor allocation trajectory of country A by depicting the points $\mathbf{x}(t_0)$, $\mathbf{x}(t_1)$,…$\mathbf{x}(t_m)$ on the standard 2-simplex and connecting them (while preserving their timely order) by line segments. We indicate the direction of movement (i.e., the timely order of the points) along the trajectory by an arrow at the last observation point. We apply this procedure to all the countries from our samples and depict the trajectories of all countries from the respective sample on one and the same simplex. In this way, we can not only observe self-intersections but also mutual intersections between countries' trajectories.

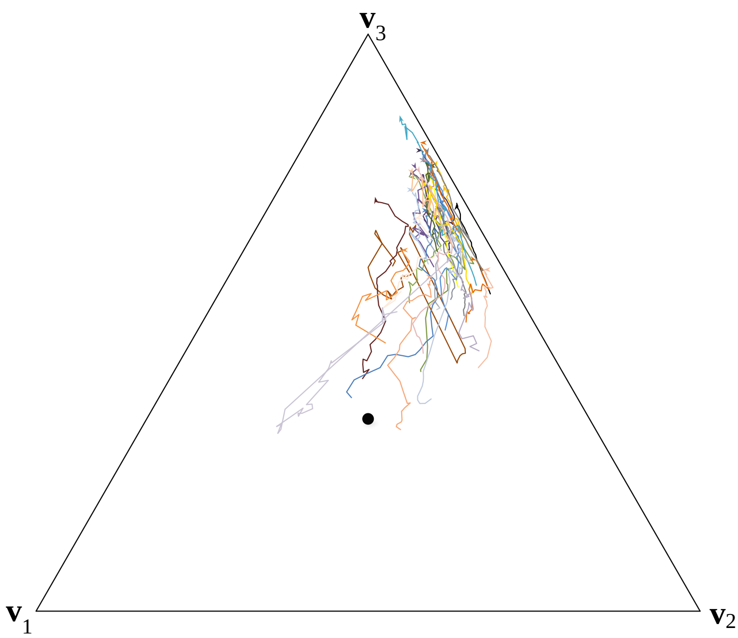
In Figures 3-5, we depict the data on the long-run labor allocation dynamics in the OECD countries on the standard 2-simplex, where the simplex refers to the employment shares of agriculture (x_1), manufacturing (x_2), and services (x_3) and the vertices ($\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$) are given by (3) (cf. Figure 1). For better visibility, Figure 5 depicts the enlarged segment of Figure 4 containing all the trajectories depicted in Figure 4. In Figures 4 and 5, we omit the arrows indicating the direction of movement along the trajectories in many cases, since they are not relevant for our discussion of the data, reflecting the topological nature of the topic.

Figure 3. *Labor allocation trajectories for USA, France, Germany, Netherlands, UK, Japan, China, and Russia.*



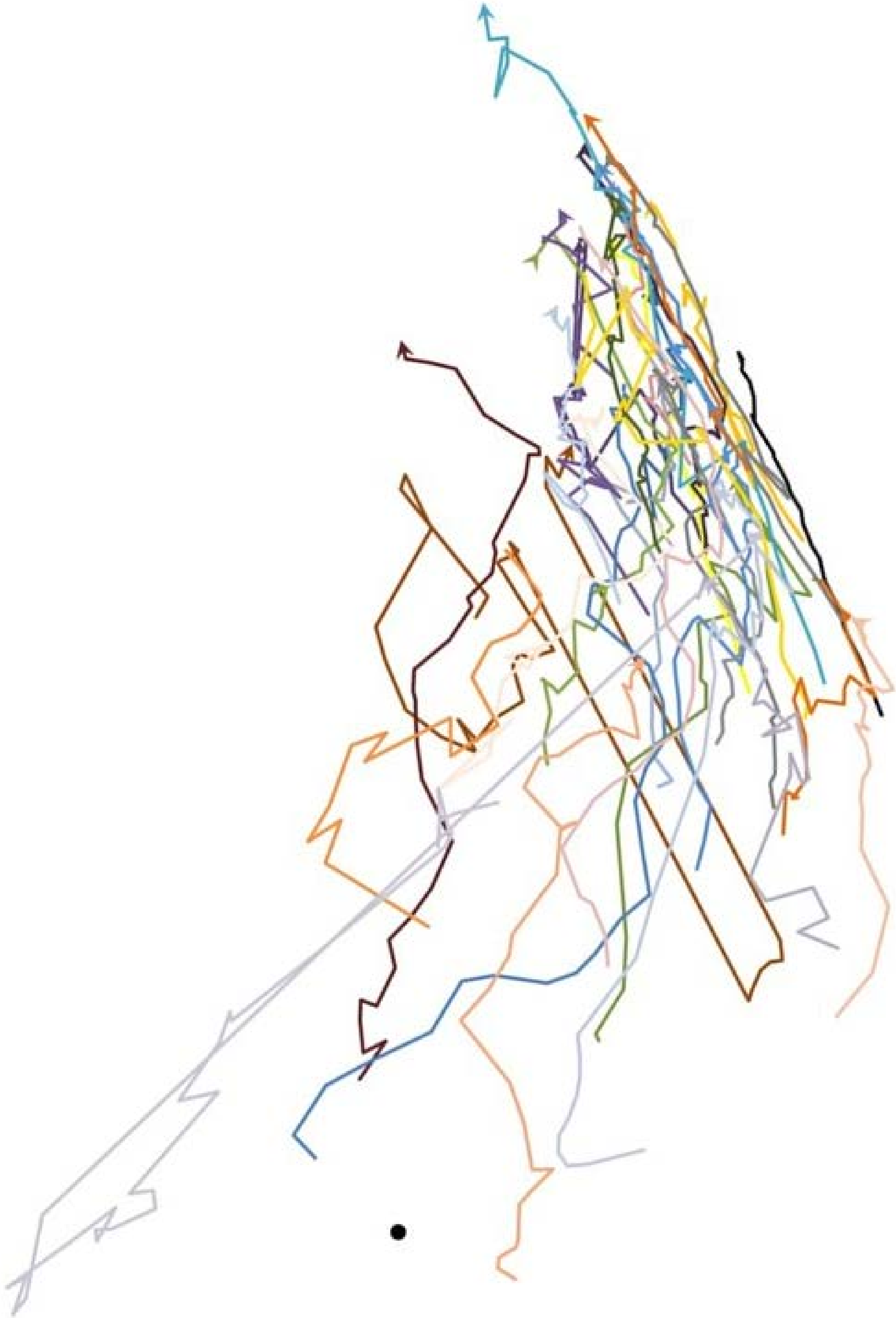
Notes: *Data source: Maddison (1995). The black dot represents the barycenter of the simplex. Abbreviations (the numbers in parentheses indicate the years for which the labor allocation points are depicted): C – China (1950, 1992), F – France (1870, 1913, 1950, 1992), G – Germany (1870, 1913, 1950, 1992), J – Japan (1913, 1950, 1992), N – Netherlands (1870, 1913, 1950, 1992), R – Russia (1950, 1992), US – United States (1820, 1870, 1913, 1950, 1992), UK – United Kingdom (1820, 1870, 1913, 1950, 1992).*

Figure 4. *Labor allocation trajectories of OECD countries over the 1980ies, 1990ies, 2000s, and 2010s.*



Notes: *Data source: The World Bank, World Databank. The black dot represents the barycenter of the simplex.*

Figure 5. *The labor allocation trajectories depicted in Figure 4 enlarged.*



Notes: *The black dot represents the barycenter of the simplex. The edges of the simplex are not visible in Figure 5. Arrows indicating the direction of movement along the trajectories are omitted in the most cases for reasons of clarity of representation.*

Figure 3 depicts the data on labor reallocation over very long periods of time (ranging from 1820 to 1992). As we can see, the trajectories of the countries *intersect*. We can observe intersections of the trajectories of the following countries: (a) Germany and UK, (b) US and France, (c) Netherlands and France, (d) US and France, (e) Netherlands and US, (f) China and US, (g) Russia and France, (h) Russia and Netherlands, (i) Japan and France, (j) Japan and Netherlands, and (k) Japan and US. Moreover, we cannot identify any *self-intersections* in Figure 3.

Figures 4 and 5 present higher-frequency data. As we can see, this data reveals again numerous intersections, thus, confirming the results derived from Figure 3. Moreover, the high-frequency data presented in Figures 4 and 5 shows many (*short-run*) self-intersections. For example, the trajectories of the following countries self-intersect: Australia, Belgium, Chile, Ireland, Island, Latvia, Luxemburg, New Zealand, Norway, Slovakia, Slovenia, Suisse, Sweden, and Turkey. We cannot observe any *longer-run* self-intersections, e.g., large trajectory loops (covering long time periods).

The observations discussed in this section are summarized by Stylized Facts 1 and 2.

Stylized Fact 1. *The labor allocation trajectories of different countries intersect mutually (in the long run).*

Stylized Fact 2. a) *The long-run dynamics of labor allocation can be represented by non-self-intersecting trajectories.* ***b)*** *Only short-run intersections are observable in the data, i.e., there are no long-run trajectory loops.*

For further evidence on Stylized Facts 1 and 2, see Stijepic (2017e).

4. Toward a theoretical explanation of the observed topological properties of structural change paths

4.1 Toward a theoretical explanation of intersection of trajectories (Stylized Fact 1)

In this section, we discuss how the (self-)intersection of trajectories can be explained by structural change models that are representable by differential equation systems. Most structural change models are representable by differential equations, since the typical long-run modeling assumptions rely on smooth and differentiable (production and utility) functions; for example, all the models listed in Footnote 1 are based on continuous and differentiable functions (with respect to time).

We start the discussion by recapitulating the well-known result from differential equation theory stating that smooth autonomous differential equation systems generate only non-(self-)intersecting trajectories for given/constant system parameters. For references, see, e.g., Stijepic (2015, p.84f.) and Stijepic (2017c). For example, assume that a structural change model (that is consistent with Definitions 1 and 2) can be represented by the following initial value problem:

(6) $\forall t \in D \subseteq R\ \forall \mathbf{x}_0 \in U \subseteq R^3\ \forall \mathbf{p} \in P\ d\mathbf{x}(t)/dt = \Phi(\mathbf{x}(t),\mathbf{p}),\ \mathbf{x}(0) = \mathbf{x}_0,\ 0 \in D$

where $\mathbf{p}$ is a parameter vector taking values in the set P. There exists a *unique solution* of (6) if the function Φ has certain (smoothness) characteristics[3] (for $\mathbf{p} \in P$). In this case, for a given parameter value $\mathbf{p} \in P$, the differential equation system (6) generates a *family of continuous, non-intersecting and non-self-intersecting trajectories* (where each trajectory corresponds to a different initial value $\mathbf{x}_0 \in U$). Thus, a structural change model that generates a smooth system of the type (6) is consistent with Stylized Fact 2. However, if we assume that each trajectory generated by (6) corresponds to a different country (i.e., if countries differ by initial states $\mathbf{x}_0$), there is no intersection between countries' trajectories.[4] Thus, Stylized Fact 1 is violated. Note that the empirical evidence, e.g., Figures 3-5, implies that the initial states of countries differ (at least if we choose an initial time point within the last 150 years or so, which is standard in structural change modeling).

(Self-)intersections can be generated if we depart from the assumptions made regarding system (6). In particular, a differential equation system can generate a family of (self-)intersecting trajectories if the system is non-autonomous (*Case A*), non-smooth (*Case B*), or characterized by parameter perturbations (*Case C*).[5] In these cases, (6) can generate intersecting trajectories and, thus, can be consistent with Stylized Fact 1 if we assign to each country a different trajectory (i.e., a different initial state) of the system. This fact can be easily proven by, e.g., finding examples of non-autonomous, non-smooth or perturbed differential equation systems that generate (self-)intersections.

[3] The mathematical literature discusses different sets of conditions that ensure the "uniqueness of solutions" (for a given parameter setting $\mathbf{p}$). In general, these conditions require that the function Φ (cf. (6)) is smooth in some sense (for a given parameter setting $\mathbf{p}$). For an overview of these conditions, see Stijepic (2015, p.84f.).

[4] We assume here that different countries are modelled by one and the same model (i.e., by (6)) and that countries differ by initial states (cf. Appendix A.2.1). However, there are alternative ways to model the dynamics of a group of countries by using structural change models and differential equation systems, as discussed in Appendix A.2.

[5] In our meta-model represented by (6), Case A can be implemented by replacing the function $\Phi(\mathbf{x}(t),\mathbf{p})$ by a function $\Gamma(\mathbf{x}(t),\mathbf{p},t)$, Case B corresponds to the assumption that Φ is non-smooth or non-continuous, and Case C implies that the parameter vector $\mathbf{p}$ changes at least one time.

Moreover, obviously, intersections between countries' trajectories *can* arise if we model each country's structural path by using a different model (*Case D*), where each model generates a different differential equation system (cf. Appendix A.2.3), or assume that different countries have different parameter vectors **p** (*Case E*; cf. Appendix A.2.2).[6]

In Appendix B, we demonstrate briefly how Cases C and E give rise to (self-)intersections in the Kongsamut et al. (2001) model, which is one of the standard structural change models. This demonstration elucidates that Cases C and E (and A, B, and D) can be used to analyze whether the existing/standard structural change models can explain the observed intersections of countries' trajectories (Stylized Fact 2) given the observed values/dynamics/variations of the parameters (**p**) of these models. This question is far beyond the scope of our paper, since it seems to require extensive work and discussion of data (on parameters **p**) and econometric techniques. Yet it seems a very interesting topic for further research.

Overall, this section implies that the observation of intersections of countries' trajectories is not surprising from the mathematical-theoretical point of view. Smooth autonomous differential equation systems generating non-intersecting trajectories can be regarded as special cases of dynamic systems (or mathematical anomalies), and intersections of countries' trajectories can arise even if structural change is modelled by such systems (cf. Cases C-E). Moreover, in the light of (a) observable cross-country heterogeneity regarding technologies and preferences[7] and (b) the ceteris paribus nature of economic laws, it makes sense to assume (a priori) that cross-country variation in parameters **p** (cf. Case E) is an explication of the intersection of countries' trajectories (among others).

4.2 Toward a theoretical explanation of non-self-intersection (Stylized Fact 2)

As discussed by Stijepic (2015), the standard structural change models (cf. Footnote 1) generate non-self-intersecting trajectories; thus, each of the models can be regarded as an (*implicit*) theoretical explanation of non-self-intersection. However, none of the previous contributions seeks to explain or mentions non-self-intersection explicitly; moreover, the assumption sets of the models differ significantly such that it is difficult to understand the *common* theoretical rationale for non-self-intersection by superficially analyzing these models. Thus, first, we take a brief look on how non-self-intersection is achieved in these models (cf. Section 4.2.1) and, then, briefly discuss a theoretical rationale for non-self-

[6] For example, if we model the dynamics of two countries (country A and country B) by using our meta-model (6), Case E can be modelled by assuming that country A is characterized by vector $\mathbf{p}_A \in P$ and country B is characterized by vector $\mathbf{p}_B \in P$ where $\mathbf{p}_A \neq \mathbf{p}_B$.

[7] This heterogeneity becomes most evident when comparing developed and underdeveloped countries.

intersection, where self-intersection is more or less *explicitly* considered by a utility-maximizing representative household (cf. Section 4.2.2).

4.2.1 Implicit (partial) theoretical explanations by the previous literature

In general, standard structural change models (cf. Footnote 1) can be represented by the following *metal-model*:

(7) $\mathbf{x}(t) = \Psi(\mathbf{a}(t),\mathbf{z}(t))$ for $t \in [0,\infty)$

where $\mathbf{x}(t)$ represents the employment shares (cf. Definition 1), $\mathbf{a}(t) \equiv (a_1(t), a_2(t),\ldots a_m(t)) \in R^m$ is the vector of time dependent exogenous parameters and $\mathbf{z}(t) \equiv (z_1(t), z_2(t),\ldots z_m(t)) \in R^n$ is the vector of endogenous and time dependent variables, i.e., non-constant variables that are explained within the model. The vector $\mathbf{z}$ does not contain the employment shares $\mathbf{x}$. In some sense, (7) may be understood as a solution of a (non-autonomous) differential equation system.

In most structural change and growth models, the *exogenous parameters* $\mathbf{a}(t)$ represent population and (sectoral) technology parameters and it is assumed that the parameters are growing/declining strictly monotonously (at constant rates), i.e., $a_i(t) = a_i^0\exp(g_i t)$, where a_i^0, $g_i \in R$ are given (and constant) for $i = 1,\ldots m$ and $t \in [0,\infty)$.

We can already see that the curve $\mathbf{a}(t)$, $t \in [0,\infty)$, generates a continuous and non-self-intersecting trajectory ($T_{\mathbf{a}} := \{\mathbf{a}(t) \in R^m: t \in [0,\infty)\}$) in m-dimensional real space (cf. Definitions 3 and 4); the curve/trajectory starts in $\mathbf{a}(0) = (a_1^0, a_2^0,\ldots a_m^0)$ and converges to infinity or zero (in some dimension) for $t \to \infty$. In other words, the trajectory $T_{\mathbf{a}}$ is homeomorphic to the [0,1) interval.

In neoclassical structural change models (e.g., Kongsamut et al. (2001) and Ngai and Pissarides (2007)), the *endogenous variables* $\mathbf{z}(t)$ represent (aggregate) consumption and capital. These models have the following characteristics:

1.) The differential equation system describing the dynamics of consumption and capital is derived from the typical neoclassical theoretical microfoundation (intertemporal utility maximization in Ramsey-(1928)-Cass-(1965)-Koopmans-(1967)-type multi-sector models).

2.) It is shown that the solution of the consumption-capital differential equation system (or a transformation of it) generates a saddle path along which the economy converges to a fixed point (‘steady state’).

3.) Economic arguments[8] are provided ensuring that the economy is always placed on one of the two stable arms of the saddle path, which we name here T_{ck1} and T_{ck2}. Thus, for all (empirically relevant) initial conditions, the economy is located on either T_{ck1} or T_{ck2} and converges along one of these stable arms to the fixed point. The stable arms are continuous and non-(self-)intersecting trajectories in the sense of Definitions 3 and 4 and are, thus, homeomorphisms of the [0,1) or (0,1) interval.

Overall, the dynamics of the employment shares $\mathbf{x}$ in standard structural change models are dependent on exogenous ($\mathbf{a}$) and endogenous ($\mathbf{z}$) variables, which are describable by non-self-intersecting curves. The mapping/function Ψ (cf. (7)), which relates $\mathbf{x}$ to $\mathbf{a}$ and $\mathbf{z}$ in these models, is a homeomorphism such that the trajectory $T_{\mathbf{x}} := \{\mathbf{x}(t)\in R^3\text{: } t\in[0,\infty)\}$ is non-self-intersecting as well. The theoretical foundations of this homeomorphism differ across models and depend on many assumptions such that it is difficult to isolate them. Nevertheless, the fact that the exogenous and endogenous variables ($\mathbf{a}$ and $\mathbf{z}$) are representable by non-self-intersecting trajectories represents a partial explanation of the non-self-intersection of the structural change trajectories in these models: if we allowed for self-intersection of $T_{\mathbf{a}}$ or T_{ck1}/T_{ck2}, then, in general, self-intersections of $T_{\mathbf{x}}$ would occur in the models covered by the meta-model of this section. Thus, the *economic theories* that ensure the non-self-intersection of $T_{\mathbf{a}}$ and T_{ck1}/T_{ck2} (and $T_{\mathbf{z}} := \{\mathbf{z}(t)\in R^n\text{: } t\in[0,\infty)\}$ in general) are partial explanations of the non-self-intersection of the structural trajectory $T_{\mathbf{x}}$. For example, the fact that the generic consumption-capital trajectories (T_{ck1} or T_{ck2}) generated by the utility maximization problem of the Ramsey-(1928)-Cass-(1965)-Koopmans-(1967) model are characterized by strictly monotonous dynamics of consumption and capital is a partial theoretical explanation of the non-self-intersection of the structural trajectory ($T_{\mathbf{x}}$) in neoclassical structural change models, as discussed in this section. Moreover, an explanation of the non-self-intersection of the exogenous parameters trajectory $T_{\mathbf{a}}$ could be searched in (multi-sector versions of) R&D-models (e.g., in Romer-(1990)-type multi-sector models such as the Meckl (2002) model) and would represent a partial theoretical explanation of the non-self-intersection of the structural trajectory $T_{\mathbf{x}}$ in standard structural change models. These topics are left for further research.

For typical examples of the models covered by the meta-model of this section, see the discussion of the Kongsamut et al. (2001) model in Appendix B (and, in particular, equation (B1)) as well as Ngai and Pissarides (2007) and, in particular, the equations '13' and '14' on p. 431 of their paper. Moreover, in Appendix C, we list other topics (e.g., savings-rate

[8] See, e.g., Barro and Sala-i-Martin (2004) for an example of such arguments.

dynamics, wealth distribution dynamics, and consumption-structure dynamics) that are covered by the meta-model of this section and characterized by non-self-intersecting structural trajectories that are partially explicable by the non-self-intersection of the trajectories of endogenous and exogenous variables **z** and **a**.

4.2.2 An explicit explanation of non-self-intersection

While in the standard literature, non-self-intersecting structural trajectories arise as a byproduct (cf. Section 4.2.1), we discuss now briefly a more direct explanation of non-self-intersection seeking to establish non-self-intersection as an economic principle by showing that a representative household tries to avoid self-intersections of the structural change trajectory if structural change is costly.

The existence of structural change costs that are borne by individuals and society (e.g., unemployment, costs of geographical relocation, environmental pollution due to industrialization, etc.; cf. Stijepic (2017b)) is well known. Obviously, it makes sense to assume that such 'costs' as unemployment and pollution enter the utility function of the representative household or social planer and that the latter seeks to minimize the magnitude of these costs, ceteris paribus. Moreover, it is obvious that some structural change paths may cause higher structural change costs than others. For example, a structural change path that is characterized by a relatively strong industrialization over the early phases of development may cause relatively high unemployment in later phases of development (see Stijepic (2017b) for a detailed discussion) and relatively high environmental pollution in general.

This discussion implies that we can assume that the representative household seeks to choose the structural change path that minimizes the structural change costs, ceteris paribus. However, the objective of structural change cost minimization may interfere with other objectives of the household. For example, a structural path corresponding to an optimal consumption program may interfere with structural change costs minimization: if a country is relatively underdeveloped, an optimal consumption program may require gradually increasing the share of the manufacturing sector (as implied by the theoretical models and evidence listed in Footnote 1); this objective may interfere the structural change cost minimization objective, since manufacturing sector growth may be associated with increasing pollution. Since the discussion of such interferences seems quite complex and lengthy (and an interesting question for further research), we focus on the following simple problem, which may serve as a theoretical benchmark and relates non-self-intersection and optimality/efficiency.

Assume that the representative household seeks to choose the structural change path that minimizes the cumulative magnitude of the structural change costs over the planning horizon (h), i.e.,

$$(8) \qquad \min \int_0^h q(t)dt$$

where $q(t)$ denotes the structural change costs that arise at time t. It can be shown that for some standard structural change cost indexes $q(t)$,[9] the structural change path that minimizes the structural change costs is non-self-intersecting (see Stijepic (2017b) for a proof that focuses on the monotonicity of cost-minimal paths, which implies almost directly non-self-intersection of the cost-minimal structural change path). We focus now on an *intuitive*/economic interpretation of this result.

The above result implies that in the context of (neoclassical) long-run labor reallocation models, the non-self-intersection of trajectories can be interpreted as an *efficiency* characteristic of the economy, as explained in the following.

Assume that a trajectory intersects itself at the coordinate point **s**. The point **s** represents a certain allocation of labor as any other point on the trajectory (on the simplex). Self-intersection of the trajectory means that the economy is at two points of time in point **s**: the first time (say at t = 1) when it traverses **s** and the second time (say at t = 2) when it intersects itself. In other words: first, the economy realizes the labor allocation **s** at t = 1; then, it deviates from this allocation over the time interval (1,2), i.e., the economy reallocates labor across sectors; finally (at t = 2), the economy returns to the allocation **s** again. (Of course, later, i.e., for t > 2, the economy may leave **s** again.) The assumption of structural change costs (q) implies that deviating from **s** over the time interval (1,2) and, thus, accumulating structural change costs and, then, returning to **s** seems to be *inefficient*, since the same end-result can be achieved by staying in **s** over the time interval (1,2), which is not associated with any structural change costs. That is, with respect to structural change costs minimization, self-intersection seems to be inferior to staying in **s** (where the latter is not defined as self-intersection according to Definition 4).

Of course, deviations from **s** over the time interval (1,2) may be optimal if some shocks lead to transitory changes in technology and preferences parameters. However, in general, growth theory abstracts from such 'short run' shocks by assuming static utility functions (that are maximized by infinitely living perfect foresight representative households) and monotonous

[9] One very simple example of such a cost index is $q(t) := |dx_1(t)/dt| + |dx_2(t)/dt| + |dx_3(t)/dt|$, where the structural change costs q are a monotonous function of the number of workers reallocated (see Stijepic (2017b)).

sectoral technology ($\mathbf{a}(t)$) dynamics (cf. Section 4.2.1). In this case, the monotonicity of the technology variables $\mathbf{a}(t)$ in association with our 'inefficiency argument' ensures that the household chooses a monotonous (labor reallocation) path to its future destination. In other words, our 'inefficiency argument' can be regarded as a theoretical foundation of the homeomorphism (Ψ) in the meta-model of Section 4.2.1.

5. Concluding remarks

Traditionally, the structural change literature relies on the mathematical branches of calculus, analysis, and algebra. The aim of our paper is to demonstrate the applicability of topological concepts (such as self-intersection and intersection of trajectories as well as homeomorphisms) in the analysis of structural change, seeking to lay the foundations for the application of a large set of topological concepts and theorems in this field. We have demonstrated how topological characteristics can be used to study empirical evidence, classify models, compare models with evidence, and derive new theories and research topics. Since the paper is devoted to the introduction of topological methods in structural change analysis, it deals necessarily with the most basic topological concepts and methods. However, the level of methodical sophistication can be gradually increased on the basis of our discussion, as demonstrated by Stijepic (2015), who uses the non-self-intersection characteristic in structural change predictions, and Stijepic (2017c,d), who discusses the applicability of the Poincaré-Bendixson theory, which is a major topological result regarding dynamic systems in the plane, in structural change modeling.

Further research could focus on higher-dimensional problems (four- and multi-sector frameworks and, thus, with three- and higher-dimensional simplexes) and more complex theorems (relating to, e.g., structural stability and occurrence of chaos). Moreover, while we used our method to analyze the Kongsamut et al. (2001) model in Appendix B, many other standard models (e.g., the Acemoglu and Guerrieri (2008) model) can be analyzed upon their topological properties and their consistency with the empirical facts. This analysis can go much further than the analysis in our paper, which was limited by space restrictions and the necessity to lay the foundations of our approach. For example, each structural change model from the previous literature can be analyzed (on the basis of the results of Section 4.1) upon two questions: (1.) which exogenous model parameters must be varied and how must they be varied to generate (self-)intersections of the structural trajectories in the model; (2.) did such parameter variations occur in the countries that experienced (self-)intersections. Depending on the answers to these questions, model critique can be formulated and new model classes may become necessary. Another way of extending our research relates to the topics covered by it. While Definitions 1 and 2 are relating to labor allocation, many other topics (e.g., savings rate dynamics, functional income distribution, consumption structure dynamics, and personal income distribution) can be studied by using the methods discussed in our paper (see Appendix D for a generalization of Definitions 1 and 2 and an extensive discussion of topics covered by these definitions and the topological approach). Furthermore, it could be interesting to continue the discussion started in Section 4.2.2 and develop further explanations of non-self-intersection. Overall, it seems that our approach generates a huge set of new research topics. These are left for further research.

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APPENDIX

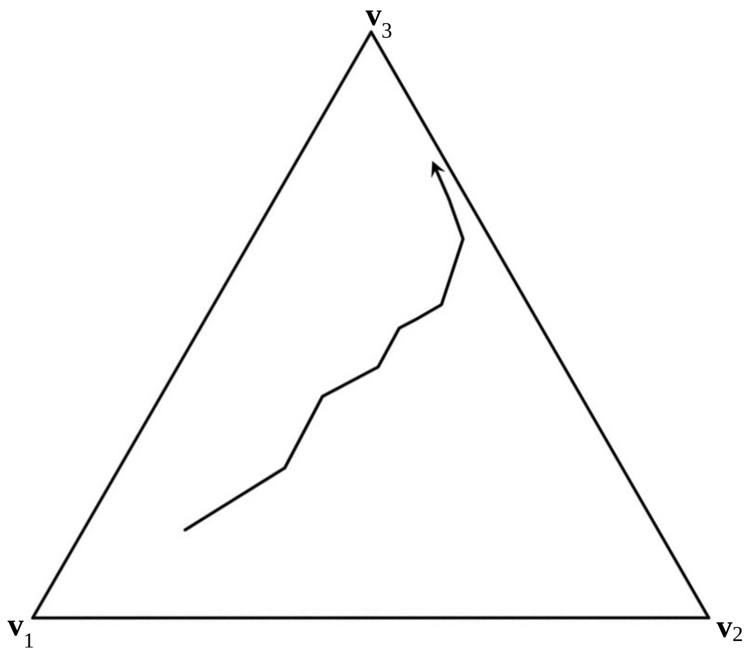
Appendix A. On the explanation of empirical observations by structural change models

In this section, we discuss how the structural dynamics of a country or a group of countries can be explained by using the meta-model (4)-(5), which covers a wide range of structural change models. This discussion does not refer to a specific empirically observed characteristic of structural change trajectories; it is rather of methodological character. Section A.1 deals with the question of how to explain the dynamics of *one country* by using a structural change model. While the answer to this question is quite obvious, there are different ways of explaining the dynamics of a *group of countries* by using a structural change model (cf. Section A.2). As we will see in Section A.2.4, these ways reflect different (methodological) notions of economic law underlying the structural change models.

A.1 Explanation of a country's dynamics

Assume that we have data on the dynamics of labor allocation over some period of time (e.g., 1820-2003) in a country (e.g., the US). Furthermore, assume that we construct this country's structural trajectory on the simplex by using this data (cf. Section 3). Figure A1 depicts an example of such a trajectory.

Figure A1. *The trajectory of labor allocation across agriculture, manufacturing, and services in USA covering the period 1820-2003.*



Notes: *Data source: Maddison (2007). See Section 3 for method description.*

Assume now that we would like to have a theoretical explanation of the dynamics depicted by the trajectory (in Figure A1). To do so, we can choose an existing structural change model (e.g., the Kongsamut et al. (2001) model) and analyze, first, whether the model can explain (certain characteristics of) the observed trajectory. This can be done as follows. First, solve the model equations and obtain in this way a family of functions of the type (4). Note that for a given parameter vector **p**, (4) implies a family (I) of trajectories corresponding to different initial values of the system/economy (cf. (5)). Thus, among the family members (I), we must choose the trajectory that goes through the empirically observed initial state[10] of the (US) economy. Second, choose the model parameters **p** such that the model trajectory corresponding to the observed initial state of the country is as *similar*[11] as possible to the empirically observed trajectory of the country. Here, the term 'similar' may refer to *qualitative aspects*, e.g., the shape and orientation of the trajectory on the simplex, or *quantitative aspects*, where the latter refer to the question whether the model generates changes in the structure that are of similar (numerical) magnitude as the changes observed in reality for the given initial value of the country considered.

That is, to analyze whether the model can explain (certain characteristics of) the empirically observed structural trajectory of a country, we compare the (most suitable) trajectory generated by the model and the empirically observed trajectory of the country. If the model trajectory is sufficiently similar to the observed trajectory, we can say (under many restrictions) that the model is a theoretical explanation of the country's dynamics.

A.2 Explanation of the dynamics of a group of countries and relation to economic laws

Now, assume that we depict the empirically observed trajectories of different countries (e.g., OECD countries) on one and the same simplex (see, e.g., Figure 3) and aim to provide a joint explanation for the dynamics of these countries by using a structural change model (e.g., the Kongsamut et al. (2001) model). Since the empirically observed structural dynamics and, thus, the trajectories of the countries differ significantly (cf., e.g., Figure 3), we cannot explain the dynamics of all countries by only one model trajectory. That is, we need a model that generates multiple trajectories that differ from each other. The meta-model (4)-(5) implies three approaches for generating multiple/different trajectories in a model.

[10] The initial state of the country may refer to the earliest data point in the sample of structures observed for the country.

[11] Note that many parameters of structural change models cannot be observed in reality. Thus, given the theoretical/intuitive restrictions on the parameters, it may make sense to set the model parameters such that that the model fits the data best.

A.2.1 Approach 1

As implied by (5), the dynamic system (4) generates a family (of different) trajectories for a given parameter setting ($\mathbf{p}$), where each trajectory corresponds to a different initial value of the system. Thus, to model cross-country heterogeneity regarding trajectories, we can assume that (a) all the countries have the same parameter values, i.e., the parameter vector ($\mathbf{p}$) does not differ across countries, and (b) the countries differ by initial conditions. In this case, the countries belong to the same family (I) of trajectories, where each $\mathbf{x}_0 \in I$ represents a country and, in particular, a different initial condition. Example A1 may elucidate these explanations.

Example A1 (Approach 1). Assume that we aim to explain the dynamics of US, UK, and Japan by using a model that generates a trajectory family of the type (4)-(5). It is possible to assign (qualitatively and quantitatively) different trajectories of this model to the different countries if we assume that the dynamics of US, UK, and Japan can be described by (4a)/(4b)/(5) and choose the function $\phi(t,\mathbf{p},\mathbf{x}_0)$ for US, $\phi(t,\mathbf{p},\underline{\mathbf{x}}_0)$ for UK, and $\phi(t,\mathbf{p},\underline{\underline{\mathbf{x}}}_0)$ for Japan, where $\mathbf{p} \in P$, $\mathbf{x}_0, \underline{\mathbf{x}}_0, \underline{\underline{\mathbf{x}}}_0 \in I$ and $\mathbf{x}_0 \neq \underline{\mathbf{x}}_0 \neq \underline{\underline{\mathbf{x}}}_0 \neq \mathbf{x}_0$. As we can see, the initial states differ across countries, whereas $\mathbf{p}$ is the same for all countries.

In Section 4.1, we argue that (empirically observed) intersections of trajectories representing different countries cannot be explained by (6) if Approach 1 is applied (and (6) is sufficiently smooth).

A.2.2 Approach 2

As implied by (5), cross-country differences in (qualitative and quantitative) trajectory characteristics can arise if we assume that parameter values $\mathbf{p}$ differ across countries. In this case, cross-country differences in initial conditions are not necessary to create heterogeneous trajectories within a model (although due to empirical evidence, it may be reasonable to assume that cross-country differences in initial conditions exist). In other words, Approach 2 assumes that all countries have the same initial state $\mathbf{x}_0$ (cf. (4c)), but differ by parameters $\mathbf{p}$. Example A2 elucidates Approach 2.

Example A2 (Approach 2). Assume that we aim to explain the dynamics of US, UK, and Japan by using a model that generates the trajectory family (4)-(5). It is possible to assign (qualitatively and quantitatively) different trajectories of this model to the different countries if we assume that the dynamics of US, UK, and Japan can be described by (4a)/(4b)/(5) and choose the function $\phi(t,\mathbf{p}_A,\mathbf{x}_0)$ for US, $\phi(t,\mathbf{p}_B,\mathbf{x}_0)$ for UK, and $\phi(t,\mathbf{p}_C,\mathbf{x}_0)$ for Japan, where $\mathbf{x}_0 \in I$, $\mathbf{p}_A,\mathbf{p}_B,\mathbf{p}_C \in P$ and $\mathbf{p}_A \neq \mathbf{p}_B \neq \mathbf{p}_C \neq \mathbf{p}_A$. As we can see, the parameter values ($\mathbf{p}_A$, $\mathbf{p}_B$, $\mathbf{p}_C$) differ across countries, whereas the initial state $\mathbf{x}_0$ is the same for all countries.

Approach 2 corresponds to the Case E and seems useful for explaining the structural change evidence when relying on standard structural change models (cf. Section 4.1).

A.2.3 Approach 3

Approaches 1 and 2 refer to the explanation of structural change in different countries by using only *one* structural change model, e.g., the Kongsamut et al. (2001) model. A third approach could be developed by going beyond initial condition differences (Approach 1) and parameter differences (Approach 2) and assuming that each country follows its own model. This may make sense when the structural change determinants differ strongly across countries such that, e.g., US structural change is best described/explained by the Kongsamut et al. (2001) model and UK structural change is best described/explained by the Ngai and Pissarides (2007) model. We can express such model differences by using the mathematical formalism introduced in Section 2 as follows. By referring to our US-UK example, assume that US structural change is described by the system (4) and UK structural change is described by the system

(4a') $\varphi: D \times Q \times I \rightarrow S$

(4b') $\varphi: (t,\mathbf{q},\mathbf{x}_0) \rightarrow \mathbf{x} \equiv (x_1, x_2, x_3)$

(4c') $\mathbf{q} \in Q$

That is, the UK and US systems follow different functional forms (ϕ vs. φ) and depend on different parameter vector spaces ($\mathbf{p}$ vs. $\mathbf{q}$).

Approach 3 corresponds to Case D (cf. Section 4.1). Three aspects of Approach 3 are noteworthy.

First, very strong differences in economic assumptions can be represented as differences in model parameters (Approach 2). Recall that the changes in only one parameter value (e.g., the elasticity of substitution) in economic models can cause very strong changes in economic assumptions (e.g., Leontief-type vs. Cobb-Douglas-type utility/production function).

Second, in many cases, it is possible to generate meta-models that cover many different models as parameter special cases. That is, in many cases, Approach 2 covers Approach 3. For example, Stijepic (2011) and Herrendorf et al. (2014) suggest (meta-)models that transform into the Kongsamut et al. (2001) model or the Ngai and Pissarides (2007) model under certain parameter constellations. That is, the latter models are special cases of the former models that arise for certain parameter values (**p**). This example proves that it is possible to cover the cases belonging to Approach 3 by Approach 2 (and 1).

Third, Approach 3 implies/presumes that the structural change models represent ‘ad hoc laws’, which may be a point of critique for methodological reasons, as discussed in Section A.2.4.

A.2.4 The relation between the three approaches and the types of economic law

The general notion of ‘a law’ as used in natural sciences (and economics) refers to a regularity that is valid/persistent across time and space. If we use this notion in economics, we would refer to a (general) economic law as a regularity that is persistent across time and countries and, thus, can be used for predicting future dynamics in different countries. More generally speaking, the existence of some sort of economic law is the basis for any prediction of economic dynamics. For a discussion of laws in economics and natural sciences, see, e.g., Jackson and Smith (2005) and Reutlinger et al. (2015).

Our discussion of Approaches 1-3 is closely related to the methodological discussion of economic models regarding the economic laws they represent.

Approach 1, assuming that one and the same model and one and the same parameter vector can explain structural change in all time periods (considered) and in all countries, corresponds to the general notion of a (natural) law, i.e., a regularity that is valid/persistent across time (‘all periods’) and space (‘all countries’).

In contrast, Approach 2 assumes that empirical observations can be explained by one and the same model, only if we allow that parameters vary across countries. Thus, Approach 2 corresponds to the view that economic models represent ‘ceteris paribus laws’. The latter are widespread in economic modeling. See Reutlinger et al. (2015) for a discussion.

Approach 3 corresponds to ‘ad hoc laws’, i.e., regularities that are sometimes applicable and sometimes not. In particular, the applicability of an ad hoc ‘law’ differs from country to country, while (in contrast to ceteris paribus laws) it is not clearly stated when the model is applicable and when not. From the methodological point of view, the models representing ‘general laws’ or ‘ceteris paribus laws’ seems preferable, since among others, such models are directly testable by empirical evidence, in contrast to ad hoc models.[12] Furthermore, in structural change modeling, 'ad hoc laws/models' seem unnecessary, since there are many similarities in structural change patterns across countries, which can be modeled as (ceteris paribus) laws. In particular, it is, therefore, possible to replace 'ad hoc laws' by 'ceteris paribus laws', where the latter can account for cross-country differences in structural change patterns, while being testable and explicitly naming the parameters that are responsible for the observable differences across countries.

For these reasons, Approaches 1 and 2 ('general law' and 'ceteris paribus law') seem to be preferable over Approach 3.

Appendix B. An application to the theoretical structural change literature

In this section, we demonstrate how to use the results of Section 4.1 to generate (self-)intersections and compare standard structural change models with the stylized facts derived in Section 3. Since this discussion tends to be lengthy as we will see, we discuss only the Kongsamut et al. (2001) model as a major example of the modern structural change modeling literature. Of course, this choice is arbitrary to some extent and we regard all the other models[13] as interesting and important contributions to structural change theory.

First, we show that the Kongsamut et al. (2001) model belongs to the smooth autonomous differential equation class discussed in Section 4.1 (cf. (6)) and, thus, for given parameter values, the Kongsamut et al. (2001) model cannot generate (self-)intersections. Therefore, we try to generate (I) trajectory intersections in this model by assuming that there are cross-country differences (cf. Section 4.1, Case E) and perturbations (cf. Section 4.1, Case C) regarding the parameters of this model and (II) trajectory self-intersections by assuming that there are parameter perturbations (cf. Section 4.1, Case C). Note that we discuss here self-intersections although the Section 3 results show that self-intersection is not a long-run phenomenon. We do this since self-intersections occur in the shorter run and, thus, it is interesting to see whether the Kongsamut et al. (2001) model can explain short-run self-intersections.

Recall that $x_1(t)$, $x_2(t)$, and $x_3(t)$ stand for the employment shares of the agricultural, manufacturing, and services sector, respectively and, thus, $\mathbf{x}(t) \equiv (x_1(t), x_2(t), x_3(t))$ represents the labor allocation at time t.

[12] It is difficult to test the validity of model assumptions if the model is only valid for one or two countries. At least, cross-country and panel data cannot be used in this case.
[13] See Footnote 1 for some literature overviews dealing with long-run labor reallocation models.

Kongsamut et al. (2001) focus on the discussion of their model in its dynamic equilibrium state, which is named ‘generalized balanced growth path’ (henceforth: GBGP). They justify their focus on the GBGP by referring to the fact that the GBGP is consistent with the empirical evidence known as ‘Kaldor-facts’, among others. The GBGP and similar types of dynamic equilibrium are widespread in the modern structural change analysis (see Stijepic (2011)).

After some calculations based on the equations provided by Kongsamut et al. (2001), we obtain the following equations describing the dynamics of labor allocation along the GBGP of the Kongsamut et al. (2001) model:

(B1a) $x_1(t) = \beta\chi + \dfrac{B_M \overline{A}}{B_A Y_0 \exp(gt)}$

(B1b) $x_2(t) = 1 - (1-\gamma)\chi$

(B1c) $x_3(t) = \theta\chi - \dfrac{B_M \overline{S}}{B_S Y_0 \exp(gt)}$

The ‘parameters’ of this differential equation system satisfy the following restrictions (when the economy is on the GBGP), as assumed by Kongsamut et al. (2001):

(B2a) $\beta + \gamma + \theta = 1$

(B2b) $B_S \overline{A} = B_A \overline{S}$

(B2c) $\beta, \gamma, \theta, g, B_A, B_M, B_S, \overline{A}, \overline{S}, Y_0 > 0$

Although we do not seek to economically interpret the equation system generated by the Kongsamut et al. (2001) model, note that (a) Y_0 represents the aggregate output (in manufacturing terms) at time t = 0, where aggregate output grows at the rate g along the GBGP, and (b) χ stands for the aggregate consumption-expenditures-to-output ratio, which is constant along the GBGP of the Kongsamut et al. (2001) model and, obviously, satisfies the following condition

(B2d) $0 < \chi < 1$

Furthermore, it makes sense to assume that the parameters of the model are such that

(B3) $\mathbf{x}(0) \in S$

Otherwise, the employment shares would be negative, which does not make sense economically.

Note that the system (B1)-(B3) can be represented by the following differential equation system satisfying the parameter conditions (B2) and (B3):

(B4a) $\forall t \;\; x_1'(t) = \beta\chi g - g x_1(t)$

(B4b) $\forall t \;\; x_2'(t) = 0$

(B4c) $\forall t \;\; x_3'(t) = -x_1'(t)$

Thus, the GBGP dynamics of the Kongsamut et al. (2001) model are representable by a linear autonomous differential equation system.

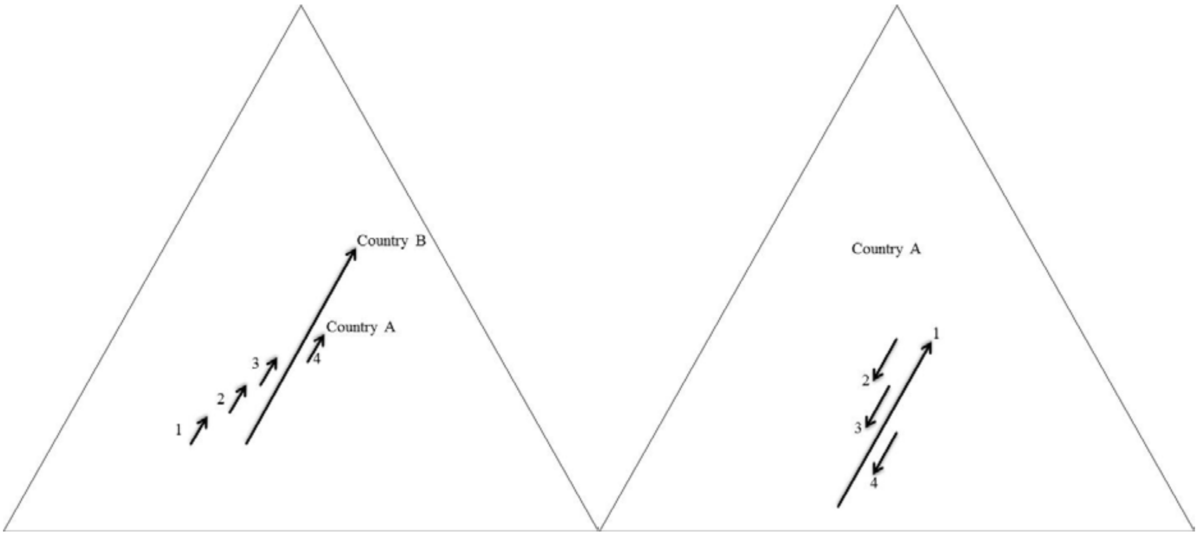
It is obvious that the system (B1)-(B3) generates a line segment on the simplex that is parallel to the $\mathbf{v}_1$-$\mathbf{v}_3$-edge of the simplex (cf. (3) and Figure 1). This is true for any parameterization of the model satisfying (B2) and for all initial conditions satisfying (B3). This fact implies that: (a) the system (B1)-(B3) belongs to the class of smooth and autonomous models discussed in Section 4.1, i.e., the system (B1)-(B3) does not generate (self-)intersections unless there is some sort of parameter variation; and (b) we cannot generate trajectory intersections by assuming Case E (cf. Section 4.1), since the countries' trajectories are always parallel (even if the parameters differ across countries).[14] However, (self-)intersections can be generated by assuming parameter perturbations, i.e., by assuming (a combination of Case E and) Case C (cf. Section 4.1). For example, (self-)intersections can be generated by assuming parameter sequences that generate the dynamics depicted in Figure B1, where the (self-)intersection occurs implicitly when the country A jumps from trajectory segment 3 to trajectory segment 4. (In empirical data, such jumps are not distinguishable from 'continuous' intersections, since the empirical data is non-continuous.)

In general, such parameter sequences seem relatively complex; models that can generate (self-)intersections by relying on simpler parameter sequences or on Case E seem preferable. However, this hypothesis cannot be discussed without econometric tests, which are beyond the scope of our paper. In general, the question of whether the complex parameter shock sequences required to generate (self-)intersections in the system (B1)-(B3) occur in reality when (self-)intersections are observed or whether other explanations (not consistent with the system (B1)-(B3)) are preferable seems interesting and is left for further research. Moreover, recall that (B1)-(B3) represents the dynamics of the Kongsamut et al. (2001) model *along the GBGP*. If we studied the economy off the GBGP, χ would not be not constant and, thus, the trajectory not linear and intersections could be possible even without the assumption of complex parameter shock sequences. We omit a detailed study of this topic, since the

[14] Note that the countries' trajectories do not overlap completely if the parameters differ across countries as assumed in Case E.

discussion above seems to be sufficient to demonstrate the applicability of our topological approach.

Figure B1. *An implicit intersection and an implicit self-intersection generated by parameter perturbations.*



Appendix C. Examples of topics and models covered by the meta-model of Section 4.2.1

If we apply the general definition of structural change introduced in Appendix D, we can show that the meta-model of Section 4.2.1 applies to a relatively heterogeneous group of core topics of growth and development theory. In particular, it can be shown that in each of the following models/topics, the non-self-intersection of the trajectory can be partially explained by the non-self-intersection of the trajectory of exogenous parameters or non-self-intersection of the trajectory of some endogenous parameters (see Stijepic (2014) for a proof):

(I) dynamics of the functional income distribution in the Solow (1956) model,

(II) savings and consumption rate dynamics in the Ramsey-(1928)/Cass-(1965)/Koopmans-(1967) model,

(III) labor reallocation across sectors in the Baumol (1967) model,

(IV) dynamics of the consumption structure in the Kongsamut et al. (2001) model,

(V) dynamics of the consumption and capital sector in the Ngai and Pissarides (2007) model,

(VI) dynamics of the personal wealth distribution in the Caselli and Ventura (2000) model.

In Appendix D, we discuss these topics in more detail.

Appendix D. A general definition of structural change and examples of topics and literature covered by it

In this appendix, we provide a more general definition of structural change and show that this definition covers a large set of topics.

Definition D1. *Let y be an aggregate index and* $y_1, y_2, \ldots y_n$ *be the components of the index, where n is a natural number. Let* $y(t)$ *and* $y_1(t), y_2(t), \ldots y_n(t)$ *denote the values of the index y and its components* $y_1, y_2, \ldots y_n$ *at time t, respectively, where* $t \in D \subseteq R$ *and R is the set of real numbers. Define* $x_i(t) := y_i(t)/y(t)$ *for all* $t \in D$ *and for all* $i \in \{1,2,\ldots n\}$. *The '**(n-dimensional) structure**' (of the index y) at time* $t \in D$ *is represented by the vector* $\mathbf{x}(t) := (x_1(t), x_2(t), \ldots x_n(t)) \in R^n$, *where* $\mathbf{x}(t)$ *satisfies the following conditions*

(D1) $\forall t \in D\ \forall i \in \{1,2,\ldots n\}\ 0 \leq x_i(t) \leq 1$

(D2) $\forall t \in D\ x_1(t) + x_2(t) + \ldots + x_n(t) = 1$.

Definition D2. *Structural change (over the period* $[a,b]$*) refers to the long-run dynamics of* $\mathbf{x}(t)$ *(over the period* $[a,b]$*; cf. Definition D1).*

In general, an n-dimensional structure (cf. Definition D1) is representable by a point on an n–1-dimensional standard simplex and, thus, structural change (cf. Definition D2) can be represented by a trajectory on this simplex.

Example D1. One of the most obvious application fields of Definition D2 is the literature on *long-run labor reallocation in multi-sector growth models*, e.g., Kongsamut et al. (2001), Ngai and Pissarides (2007), Foellmi and Zweimüller (2008), and Herrendorf et al. (2014). These models can be represented here by the following assumptions: $l_i(t)$ stands for the employment in sector i at time t, where $i = 1,2,\ldots n$; $l(t) := l_1(t) + l_2(t) + \ldots l_n(t)$ is the aggregate employment; $x_i(t) := l_i(t)/l(t)$ is the employment share of sector i at time t and, thus, $\mathbf{x}(t) \equiv (x_1(t), x_2(t), \ldots x_n(t))$ indicates the cross-sector labor allocation at time t. Obviously, these assumptions imply that the cross-sector labor allocation $\mathbf{x}(t)$ satisfies conditions (D1) and (D2) (among others since employment cannot be negative) and is, therefore, a 'structure' according to Definition D1. Finally, Definition D2 states that structural change takes place if the labor allocation $\mathbf{x}(t)$ changes in the long run. That is, structural change refers here to the long-run cross-sector labor reallocation. Thus, we have shown that the long-run labor reallocation models are covered by Definition D2.

Example D2. The three-sector framework studied in our paper is a well-known special case of Example D1. Most of the papers (e.g., Kongsamut et al. (2001), Ngai and Pissarides (2007), and Foellmi and Zweimüller (2008)) refer in some way to this framework. We obtain the three-sector framework if we assume in addition to the assumptions made in Example D1 that: $n = 3$, i.e., there are only three sectors; sector 1 ($i = 1$) represents the primary/agricultural sector, sector 2 ($i = 2$) represents the secondary/manufacturing sector, and sector 3 ($i = 3$) represents the tertiary/services sector. Then, it follows immediately that: $\mathbf{x}(t)$ represents the labor allocation across agriculture, manufacturing, and services at time t; $\mathbf{x}(t)$ is a structure, i.e., satisfies (D1) and (D2); long-run changes in $\mathbf{x}(t)$, i.e., long-run labor reallocation across agriculture, manufacturing, and services, represent(s) structural change, according to Definition D2.

Example D3. The *long-run dynamics of the savings rate* are a central topic of the neoclassical growth theory, where the Ramsey-(1928)/Cass-(1965)/Koopmans-(1967) model assumes that at every point in time t, income ($y(t)$) can only be used for savings ($s(t)$) and consumption ($c(t)$), i.e., $y(t) = s(t) + c(t)$. Let $x_1(t) := s(t)/y(t)$ denote the savings rate and $x_2(t) := c(t)/y(t)$ denote the consumption rate at time t, respectively; thus, the vector $\mathbf{x}(t) \equiv (x_1(t), x_2(t))$ indicates the savings and consumption rate. Obviously, (if we assume that there is no negative savings,) the savings-consumption rate vector $\mathbf{x}(t)$ satisfies (D1) and (D2) and, therefore, represents a 'structure' per Definition D1, where $n = 2$. Then, structural change takes place according to Definition D2 if the savings/consumption rate changes in the long run. That is, the term 'structural change' refers here to the long-run dynamics of the savings and consumption rate.

Example D4. The *long-run dynamics of the functional income distribution* play a central role in (neoclassical) growth theory. In particular, the question of whether the labor income share is constant or not is a central aspect of the discussion of the applicability of Kaldor-facts, Cobb-Douglas production functions and balanced growth paths in growth theory (see, e.g., Stijepic (2017a)). Neoclassical growth models (e.g., the Solow (1956) and the Ramsey-(1928)/Cass-(1965)/Koopmans-(1967) model) assume among others that capital and labor are the only input factors and the aggregate income is equal to the factor income. Thus, $y(t) = r(t)$

+ $w(t)$, where $y(t)$ is the aggregate income, $r(t)$ is the capital income, and $w(t)$ is the labor income at time t, respectively. In this type of model the capital income share ($x_1(t)$) and the labor income share ($x_2(t)$) are defined as follows: $x_1(t) := r(t)/y(t)$ and $x_2(t) := w(t)/y(t)$. Thus, $\mathbf{x}(t) \equiv (x_1(t), x_2(t))$ indicates the functional income distribution. It is obvious that the functional income distribution $\mathbf{x}(t)$ satisfies conditions (D1) and (D2) and, thus, is a structure per Definition D1, where $n = 2$. Structural change refers here to the long-run dynamics of the functional income distribution $\mathbf{x}(t)$, according to Definition 2.

Example D5. While the previous example refers to the dynamics of the functional income distribution, the *dynamics of personal income distribution* is covered by Definition 2 as well. (This topic is studied among others by Caselli and Ventura (2000) in the neoclassical framework.) Assume that: $y_i(t)$ stands for the income of household i, where $i = 1,2\ldots n$; $y(t) := y_1(t) + y_2(t) +...y_n(t)$ is the aggregate income; $x_i(t) := y_i(t)/y(t)$ is the share of household i in aggregate income. Thus, $\mathbf{x}(t) \equiv (x_1(t), x_2(t),\ldots x_n(t))$ represents the personal income distribution. Again, it is obvious that the personal income distribution $\mathbf{x}(t)$ satisfies conditions (D1) and (D2) and, thus, is a structure according to Definition D1. Structural change refers here to the long-run dynamics of the (discrete) income distribution $\mathbf{x}(t)$, according to Definition D2.

Example D6. The aspects of the Caselli and Ventura (2000) model that deal with the *dynamics of personal wealth distribution* can be described here as follows. $w_i(t)$ stands for the wealth of household i, where $i = 1,2\ldots n$. $w(t) := w_1(t) + w_2(t) +...w_n(t)$ is the aggregate wealth. $x_i(t) := w_i(t)/w(t)$ is the share of aggregate wealth possessed by household i. It is obvious that the personal wealth distribution $\mathbf{x}(t) \equiv (x_1(t), x_2(t),\ldots x_n(t))$ satisfies conditions (D1) and (D2) and, thus, is a structure according to Definition D1. Structural change refers here to the long-run dynamics of the (discrete) wealth distribution $\mathbf{x}(t)$.

Example D7. The dynamics of the consumption and capital sector play a central role in the recent multi-sector growth modeling literature, which includes, e.g., Kongsamut et al. (2001), Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008), Herrendorf et al. (2014), and Boppart (2014). These models focus their analysis on specific dynamic equilibrium paths that are consistent with the Kaldor facts (cf., e.g., Kongsamut et al. (2001) and Stijepic (2011)). These paths have different names in the literature, e.g., ‘generalized balanced growth paths’ (cf. Kongsamut et al. (2001)), ‘aggregate balanced growth paths’ (cf. Ngai and Pissarides

(2007)), and 'constant growth paths' (cf. Acemoglu and Guerrieri (2008)). Nevertheless, they have a common characteristic: they exist only if the dynamics of the consumption and capital sector are balanced among others (cf. Stijepic (2011)). Thus, the discussion of the structural change relating to the capital-consumption structure is a central aspect of the modern multi-sector growth literature. This structure can be described here as follows. Assume that $c(t)$ is the value of consumption (i.e., the value of the output of the consumption sector), $dk(t)$ is the value of investment (i.e., the value of the output of the capital sector), and $y(t) := c(t) + dk(t)$ is the value of aggregate output at time t, respectively. Define $x_1(t) := c(t)/y(t)$ and $x_2(t) := dk(t)/y(t)$; thus, $\mathbf{x}(t) \equiv (x_1(t), x_2(t))$ indicates the consumption-capital structure at time t. It is obvious that the consumption-capital structure $\mathbf{x}(t)$ satisfies (D1) and (D2) and is, thus, a structure according to Definition D1, where $n = 2$. Structural change refers here to the long-run change in the capital-consumption structure $\mathbf{x}(t)$, according to Definition D2.

Example D8. The dynamics of the consumption structure play a central role in the multi-sector literature discussed in Examples D1 and D6 (cf., e.g., Kongsamut et al. (2001) and Boppart (2014)). These dynamics can be studied as follows. Let $x_i := c_i(t)/c(t)$ denote the consumption share of sector i at time t for $i = 1,2,\dots n$, where $c_i(t)$ stands for the consumption expenditures on goods/services produced by sector i at time t and $c(t) := c_1(t) + c_2(t) +\dots c_n(t)$ stands for the aggregate consumption expenditures at time t. It is then obvious that $\mathbf{x}(t) \equiv (x_1(t), x_2(t),\dots x_n(t))$, which indicates the consumption structure of the economy at time t, satisfies (D1) and (D2) and, thus, represents a structure according to Definition D1. Furthermore, structural change takes place according to Definition D2 if the consumption shares change in the long run. That is, structural change refers here to the long-run changes in the consumption structure.