Consistent Modeling of Risk Averse Behavior with Spectral Risk Measures

Hans Peter Wächter und Thomas Mazzoni

Diskussionsbeitrag Nr. 455
August 2010

Diskussionsbeiträge der Fakultät für Wirtschaftswissenschaft
der FernUniversität in Hagen

Herausgegeben vom Dekan der Fakultät

Consistent Modeling of Risk Averse Behavior with Spectral Risk Measures

Hans Peter Wächter and Thomas Mazzoni*

August 11, 2010

Abstract

This paper introduces a new method for modeling risk averse behavior with spectral risk measures. It is shown that recent approaches, using phenomenological correspondences or results from robust statistics, generally do not generate consistent results.
Our method is based on the dual theory of choice. We show that it is possible to encode preference relations in distorted probability measures, which themselves induce admissible spectral risk measures. This way, risk averse behavior can be mapped onto the risk spectrum defining a spectral risk measure and can be quantified using a local Pratt-Arrow-like coefficient.

Keywords: Spectral risk measures; Decision theory; Risk aversion; Coherent risk measure; Pratt-Arrow-coefficient

1 Introduction

Defining "risk" as the uncertainty of future returns of some financial position or portfolio, the measurement of risk becomes a major concern of any institution dealing with financial products. Financial science has formalized this uncertainty by considering the random distribution of future returns (the so called P[rofit]&L[oss]-distribution), and consequently, has given a formalized measure of risk in terms of a functional of the P&L-distribution, i.e. a mapping ρ from the space of random returns into the real numbers. The resulting real number represents the quantity of risk inherent in a given P&L-distribution.

Within the theory of risk measures, two different perspectives can be coarsely distinguished. On the one hand, minimal requirements for risk measures were formulated with respect to the measuring process. This aspect is emphasized by the comparison of risk and temperature measuring processes in Acerbi (2004, p. 147). On the other hand, risk measures were used to model risk aversion

*Thomas Mazzoni, Department for Applied Statistics, University of Hagen, Germany, Tel.: 0049 2331 9872106, Email: Thomas.Mazzoni@FernUni-Hagen.de

of a decision maker in order to rationalize the decision process with respect to different (financial) positions. This perspective is common in the field of risk-reward models, dating back to Markowitz' seminal theory of portfolio selection (Markowitz, 1952, 1959).

The definition and investigation of spectral risk measures by Acerbi (2002), in conjunction with their representation as weighted integral over quantiles of the entire P&L-distribution, has recently triggered considerable research effort, leading to some understanding of risk aversion in terms of weighting functions in the integral kernel of spectral risk measures. However, a systematic connection between decision theory and spectral risk measures, which is needed to consistently model risk averse behavior using spectral risk measures is still missing.

This triggered discussions in recent papers on how to relate classical expected utility theory with spectral risk measures or more specifically, how to relate the utility function with the weighting function defining a spectral risk measure. Dowd et al. (2008) and Tao et al. (2009) favor a purely phenomenological approach by choosing a utility function and simply assigning a weighting function of the same type to the spectral risk measure. Dowd et al. found that spectral risk measures, constructed in this way, not necessarily behave consistently with the expectation of the chosen utility function. They interpreted this result as systematic shortcoming and raised serious criticism on the concept of spectral risk measuress. Sriboonchitta et al. (2010) used results of robust statistics to establish a formalized scheme for the direct calculation of weighting functions from given utility functions. However, they did not provide nontrivial examples to demonstrate the significance of their method.

The aim of this paper is to derive a systematic relation between the class of spectral risk measures and decision theory in order to consistently model risk averse behavior with spectral risk measures or conversely in order to be able to interpret spectral risk measures consistently within the framework of decision theory. To accomplish this task, we first review the results of Dowd et al. (2008). In particular, we show that their objections on the possibility of modeling risk averse behavior with spectral risk measures is based on an artifact of their calculation. Subsequently, the scheme proposed by Sriboonchitta et al. (2010) is elaborated on nontrivial utility functions. Our results imply that this scheme does not lead to a consistent relation between utility theory and spectral risk measures. Therefore, we finally take a different approach by deriving a unique connection between spectral risk measures and the dual theory of choice, using results from actuarial literature; namely the premium principle and distortion risk measures. We demonstrate that this connection represents the missing link in providing a measure for subjective risk aversion in terms of the weighting functions defining spectral risk measures.

The remainder of the paper is organized as follows: In section 2, a brief introduction to spectral risk measures, decision theory, stochastic dominance, and risk aversion is given for later reference. In section 3, the aforementioned approaches of Dowd et al., Tao et al., and Sriboonchitta et al. are briefly reviewed and discussed. In section 4, a consistent relationship between spectral risk measures and the dual theory of choice is constructed, using results from actuarial literature. We also revisit examples of Dowd et al. (2008) to explain their inconsistencies. A brief summary is given as concluding section 5.

2 General theory

2.1 Spectral risk measures

Suppose $L^0(\Omega, \mathcal{F}, P)$ to be the space of all measurable, real-valued functions (i.e. random variables) on some probability space $(\Omega, \mathcal{F}, P)$. Further, suppose the P&L of a financial position to be determined at some future time T by the state of the world at that time, and to be fully described by some random variable $X \in L^0(\Omega, \mathcal{F}, P)$ or its cumulative distribution function (P&L-distribution) $F_X(x)$ today. In what follows, we assume that all distribution functions are sufficiently smooth. A spectral risk measure is defined as a functional $\rho : L^0(\Omega, \mathcal{F}, P) \to \mathbb{R}$ for which the following six axioms hold (Acerbi, 2004; Tasche, 2002):

- A1 For all $X, Y \in L^0(\Omega, \mathcal{F}, P)$ with $X \leq Y$ holds: $\rho(Y) \leq \rho(X)$ (monotonicity).
- A2 For all $X \in L^0(\Omega, \mathcal{F}, P)$, $\lambda \in \mathbb{R}_0^+$ and $\lambda X \in L^0(\Omega, \mathcal{F}, P)$ holds: $\rho(\lambda X) = \lambda \rho(X)$ (positive homogeneity).
- A3 For all $X \in L^0(\Omega, \mathcal{F}, P)$ and $\lambda \in \mathbb{R}$ holds: $\rho(X + \lambda) = \rho(X) - \lambda$ (translation invariance).[1]
- A4 For all $X, Y, X + Y \in L^0(\Omega, \mathcal{F}, P)$ holds: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (sub-additivity).
- A5 For all $X, Y \in L^0(\Omega, \mathcal{F}, P)$ with $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R}$ holds: $\rho(X) = \rho(Y)$ (law invariance).
- A6 For all comonotonic $X, Y \in L^0(\Omega, \mathcal{F}, P)$ holds: $\rho(X + Y) = \rho(X) + \rho(Y)$ (comonotonic additivity).

The first four items are the well-known axioms of coherent risk measures as postulated by Artzner et al. (1999). The remaining two axioms are special to spectral risk measures, which thus constitute a subset of coherent risk measures.

Spectral risk measures can be expressed as weighted integral over the quantiles of the P&L-distribution, whose risk is to be measured. Certain requirements on the weighting function ensure the validity of the axioms A1 to A6. Conversely, any weighting function, which meets these requirements, defines a spectral risk measure using the corresponding integral representation (cf. Acerbi, 2002). Specifically, any function $\varphi : [0, 1] \to \mathbb{R}$, with the properties

$$\varphi(p) \geq 0 \quad \text{(positivity)},$$
$$\int_0^1 \varphi(p)dp = 1 \quad \text{(normalization), and}$$
$$\varphi(p_1) \geq \varphi(p_2) \quad \text{(monotonicity), for any } p_1 \leq p_2$$

[1] Obviously, ρ is defined as not being translational invariant, however, the axiom is known under the misnomer "translation invariance" in financial literature.

defines a spectral risk measure

$$\rho_\varphi(X) = -\int_0^1 \varphi(p) F_X^{-1}(p) dp = -\int_0^1 \varphi(p) q_X(p) dp, \tag{1}$$

with $F_X^{-1}(p) = q_X(p)$ representing the quantile function of X (cf. Acerbi, 2002). The weighting function φ is also referred to as risk spectrum (this term is used in the remainder of this paper).

Generally, the choice of a particular weighting function cannot be motivated from the set of axioms A1 to A6. Two complementary approaches to this problem can be distinguished with respect to the two perspectives on risk measurement introduced in section 1. On the one hand, a certain weighting function may be chosen in order to obtain the best suited measuring instrument for a given position or P&L-distribution (e.g. Acerbi, 2004; Albanese and Lawi, 2004). On the other hand, the risk spectrum may be chosen in order to reflect the subjective risk propensity of some decision maker. Focusing on this perspective, the risk spectrum φ is also called risk aversion function (cf. Acerbi, 2004). As stated in the introduction, the main contribution of this paper is to derive a rigorous formalism for systemizing the latter approach.

2.2 Decision theory, stochastic dominance and risk aversion

Consider financial positions $X, Y, \ldots$ with P&L-distributions as introduced in the foregoing subsection. Decision theory describes a decision maker by his preference relation, which allows him to make one of the following statements about any two $X, Y \in L^0(\Omega, \mathcal{F}, P)$:

$$\begin{aligned} X \succ Y &: \quad X \text{ is preferred to } Y \\ X \prec Y &: \quad Y \text{ is preferred to } X \\ X \sim Y &: \quad \text{indifference between } X \text{ and } Y. \end{aligned}$$

Further assume that some function $U : L^0(\Omega, \mathcal{F}, P) \to \mathbb{R}$ exists, which is a representation of this preference relation in the sense that

$$X \succ Y \quad \Leftrightarrow \quad U(X) > U(Y) \tag{2}$$

holds for arbitrary X, Y (cf. Föllmer and Schied, 2002). The existence of a representation of this kind can be enforced by imposing restrictions (axioms) on the set of preference relations of the decision maker (cf. Puppe, 1991; Föllmer and Schied, 2002).

While classical preference relations, as introduced above, order the set of alternatives $\{X, Y, \ldots\}$ completely, stochastic dominance separates the set into two subsets: a dominated one, which can be discarded, and a non-dominated one. Within the non-dominated subset, the alternatives remain unordered and thus equivalent from the decision makers point of view. The following two principles of stochastic dominance are commonly assumed to establish a specific order, reflecting rational behavior of the decision maker (Bawa, 1975).

The P&L-distribution of X is said to dominate the P&L-distribution of Y by first-order stochastic dominance, if

$$F_X(x) \leq F_Y(x)$$

holds for all $x \in \mathbb{R}$, and being strict for at least one x. Thus, given two financial positions, a decision maker will always prefer the position, which for any given return has a higher or equal probability of exceeding this return.

The P&L-distribution of X is said to dominate the P&L-distribution of Y by second-order stochastic dominance, if

$$\int_{-\infty}^{x} F_X(\xi)d\xi \leq \int_{-\infty}^{x} F_Y(\xi)d\xi$$

holds for all $x \in \mathbb{R}$, and being strict for at least one x. A decision maker obeying this principle is risk averse in the sense that he prefers a particular P&L-distribution over all its mean-preserving spreads, i.e. financial positions which have the same expected return, but a higher probability of extreme outcomes (cf. Rothschild and Stiglitz, 1970; Bawa, 1975). This defines the notion of risk aversion of a decision maker.

3 Spectral risk measures and expected utility theory

In this section, recent studies with the ambition to construct a relationship between the risk spectrum φ and utility theory, such that

$$\rho_\varphi(X) \leq \rho_\varphi(Y) \Leftrightarrow U(X) \geq U(Y) \tag{3}$$

holds for all $X, Y \in L^0(\Omega, \mathcal{F}, P)$, are discussed. The validity of equation (3) implies that spectral risk measure and utility function (see next section) are equivalent descriptions of the same unique preference relation of a decision maker. In particular, by constructing a mapping between utility function and risk spectrum, the concept of risk aversion is operationalized consistently in the spectral risk measure framework.

In subsection 3.1, expected utility theory is reviewed and the utility function is introduced. Subsection 3.2 critically discusses the phenomenological approaches of Dowd et al. and Tao et al. with particular focus on the results of Dowd et al.. The method of Sriboonchitta et al. is presented in subsection 3.3. It is shown that the method fails in consistently connecting expected utility theory and spectral risk measures.

3.1 Expected utility theory

Expected utility theory assumes a particular representation of the preference relation (2),

$$U(X) = \int_{-\infty}^{\infty} u(x)dF_X(x) =: \mathrm{E}_u[X] \tag{4}$$

with a utility function u on the real numbers (cf. Föllmer and Schied, 2002). The basic idea is that decisions are not simply determined by the expected future outcome E[X], but that a utility $u(x)$ is assigned to any possible future outcome x (cf. Bernoulli, 1738; Raiffa, 1970). Note that the utility function is unique only up to linear transformations $au(x)+b$, with $a, b \in \mathbb{R}$ and $a > 0$.

First- and second-order stochastic dominance (subsection 2.2) can be enforced by restricting the set of admissible utility functions. Requiring $\frac{du}{dx} > 0$ ensures that first-order stochastic dominance is obeyed. Requiring $\frac{d^2u}{dx^2} \leq 0$ (concave u) ensures second-order stochastic dominance and thus risk averse behavior of the decision maker. In this context, Pratt (1964) and Arrow (1971) introduced a simple local measure of risk aversion, encoded in the utility function

$$r_{\mathrm{PA}}(x) = -\frac{d^2u}{dx^2} \Big/ \frac{du}{dx}, \tag{5}$$

e.g. Keeney and Raiffa (1993).

It can be shown that for a given preference relation, a representation of the form (4) exists if and only if the preference relation obeys a certain set of axioms (von Neumann and Morgenstern, 1947; Puppe, 1991). In particular, expected utility theory requires the so called independence axiom (Raiffa, 1970; Puppe, 1991)

IA For any $X, Y, Z \in L^0(\Omega, \mathcal{F}, P)$, $x \in \mathbb{R}$ and $\lambda \in [0,1]$ holds:
$X \succcurlyeq Y \Rightarrow \lambda F_X(x) + (1-\lambda)F_Z(x) \succcurlyeq \lambda F_Y(x) + (1-\lambda)F_Z(x)$.

3.2 Phenomenological approaches

The works of Dowd et al. (2008) and Tao et al. (2009) consider particular types of utility functions and translate them into corresponding risk spectrums of a similar type.

Dowd et al. consider the utility functions $u_1 = -e^{-kx}$ with $k > 0$ and $u_2 = x^{1-\gamma}$ with $\gamma > 0$, and assign the corresponding risk spectrums $\bar{\varphi}_k^{(1)}(p) = \frac{ke^{-k(1-p)}}{1-e^{-k}}$, $\bar{\varphi}_\gamma^{(2)}(p) = \gamma(1-p)^{\gamma-1}$ for $\gamma \in (0,1)$, and $\bar{\varphi}_\gamma^{(3)}(p) = \gamma p^{\gamma-1}$ for $\gamma > 1$, respectively[2]. Figure 1 shows the resulting spectral risk measures $\rho_{\bar{\varphi}^{(i)}}(X_j)$ for a standard normal distribution X_1 and a standard uniform distribution X_2. The calculation of $\rho_{\bar{\varphi}^{(i)}}(X_2)$ can be accomplished analytically, using the quantile function $q_{X_2}(p) = p$, which yields

$$\begin{aligned}
\rho_{\bar{\varphi}^{(1)}}(X_2) &= \int_0^1 \frac{ke^{-k(1-p)}}{1-e^{-k}} p\,dp = \left(\frac{1}{1-e^{-k}} - \frac{1}{k}\right) \\
\rho_{\bar{\varphi}^{(2)}}(X_2) &= \int_0^1 \gamma(1-p)^{\gamma-1} p\,dp = \frac{1}{\gamma+1} \\
\rho_{\bar{\varphi}^{(3)}}(X_2) &= \int_0^1 \gamma p^{\gamma-1} p\,dp = \frac{\gamma}{\gamma+1}.
\end{aligned}$$

[2]Note that Dowd et al. assign positive values to losses. In contrast to the remainder of this paper, the convention is kept in this subsection to stay closer to the original work. To indicate this, the risk spectrums carry a bar on top. The transformation from this convention to the convention of negative values for losses is described in great detail in appendix A

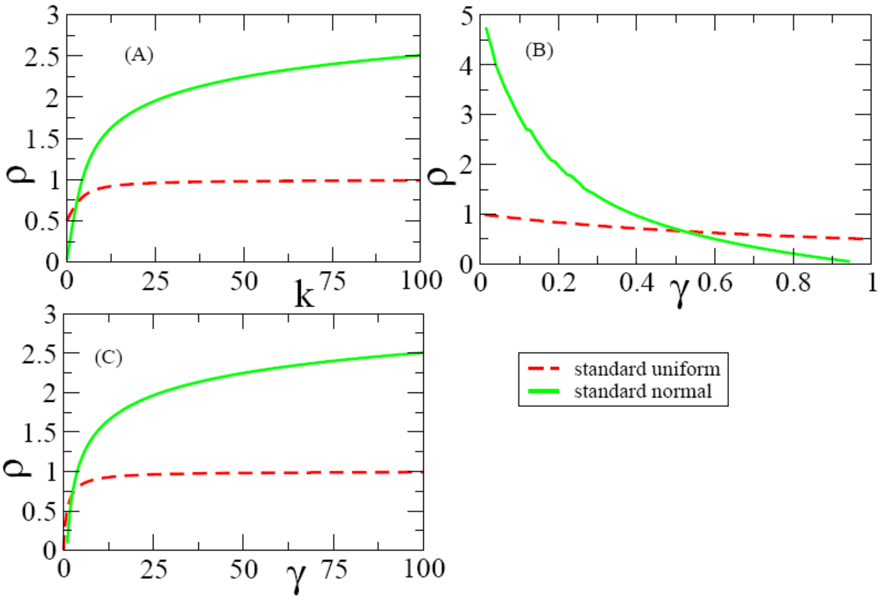

Figure 1: Spectral risk measure $\rho_{\bar{\varphi}^{(i)}}(X)$ as function of the parameter of the risk spectrum $\bar{\varphi}^{(i)}$ for X being a standard normal (solid) or a standard uniform (dashed) distribution, respectively. Plot (A): risk spectrum $\bar{\varphi}_k^{(1)}(p) = \frac{ke^{-k(1-p)}}{1-e^{-k}}$, (B): $\bar{\varphi}_\gamma^{(2)}(p) = \gamma(1-p)^{\gamma-1}$, (C): $\bar{\varphi}_\gamma^{(3)}(p) = \gamma p^{\gamma-1}$.

The calculation of $\rho_{\bar{\varphi}^{(i)}}(X_1)$ is accomplished by first drawing a sample of 10^7 normally distributed random numbers, using the Box-Muller algorithm (cf. Knuth, 2009) with a random stream of the MT19937 generator of Matsumoto and Nishimura ("Mersenne Twister", 1998). Then the integral (1) is solved numerically by Gauss-Konrod integration. The quantiles are extracted from the simulated sample by linear interpolation. The implementation was conducted using standard functions from the GNU Scientific Library (version 1.14), see Galassi et al. (2009) for a detailed reference.

The k-dependence of $\rho_{\bar{\varphi}^{(1)}}(X_{j=1,2})$, figure 1 (A), is consistent with the interpretation of k as indicator for the degree of risk aversion, because $r_{\text{AP}}(x) = k$ and $\rho_{\bar{\varphi}^{(1)}}(X_{j=1,2})$ is isotonic in k and also trends to the worst-case loss for $k \to \infty$. However, the γ-dependence of $\rho_{\bar{\varphi}^{(2)}}(X_{j=1,2})$ in figure 1 (B) is not consistent with γ as risk aversion indicator; $r_{\text{AP}}(x) = \gamma/x$, but $\rho_{\bar{\varphi}^{(2)}}(X_{j=1,2})$ is antitonic[3]. The γ-dependence of $\rho_{\bar{\varphi}^{(3)}}(X_{j=1,2})$ again is consistent, see also Tao et al. (2009).

Dowd et al. conclude correctly that ad hoc assigning a risk spectrum to a given utility function can lead to inconsistent results, which they condense into a general concern on the properties of spectral risk measures. It should however be emphasized, that the above results occur not due to conceptual shortcomings of spectral risk measures, but because of missing theoretical content in the in-

[3]Note that figure 1 (B) contradicts figure 4 in Dowd et al. (2008), which shows a maximum at finite γ for $\rho_{\bar{\varphi}^{(2)}}(X_{j=1,2})$. Because the calculations in the present paper are conducted analytically for X_2, an error in the simulations of Dowd et al. can not be ruled out. The results in figures 1 (A) and (C) are perfectly identical to those in Dowd et al. (2008).

terlink between utility and risk spectrum. An arbitrary choice of such functions and subsequent interpretation of their parameters can lead to counterintuitive results.

A further, more fundamental doubt, concerning the compatibility of spectral risk measures and decision theory is brought forward in Dowd et al. (2008) in a footnote at the very end of their paper. It is hypothesized that the sign of the derivative $\frac{d}{d\alpha}\rho_{\bar{\varphi}_\alpha}(X)$ of a spectral risk measure with an arbitrary single-parameter risk spectrum $\bar{\varphi}_\alpha(p)$ can be changed by translation of the distribution of X. This argument implies that there is no single-parameter function, on whose parameter the corresponding spectral risk measure monotonically depends, which renders any interpretation of such a parameter in terms of risk aversion impossible. The hypothesis is motivated by the observation that all resulting terms in the derivative $\frac{d}{d\alpha}\rho_{\bar{\varphi}_\alpha}(X) = \int_0^1 \frac{\partial \bar{\varphi}_\alpha}{\partial \alpha} q_X(p)dp$ are linear in $q_X(p)$. It is then concluded that one can add an arbitrary number $\lambda \in \mathbb{R}$ to the distribution of X to translate the quantiles $q_{X+\lambda}(p) = q_X(p) + \lambda$, in order to eventually change the sign of the derivative. However, the following simple calculation shows that this hypothesis is not correct. For any $\lambda \in \mathbb{R}$ holds

$$\begin{aligned}
\frac{d}{d\alpha}\rho_{\bar{\varphi}_\alpha}(X+\lambda) &= \frac{d}{d\alpha}\left[\int_0^1 \bar{\varphi}_\alpha(p) q_{X+\lambda}(p)dp\right] \\
&= \frac{d}{d\alpha}\left[\int_0^1 \bar{\varphi}_\alpha(p) q_X(p)dp + \lambda \int_0^1 \bar{\varphi}_\alpha(p)dp\right] \\
&= \frac{d}{d\alpha}\rho_{\bar{\varphi}_\alpha}(X).
\end{aligned}$$

Due to the normalization of $\bar{\varphi}_\alpha$, the second term in the brackets is constant and thus vanishes. This fact was obviously not considered in the argumentation of Dowd et al. (2008).

In summary, the phenomenological approaches discussed above are not successful in connecting spectral risk measures and utility theory. A systematic relationship in the sense of (3) has to be established in order to reflect subjective risk aversion using spectral risk measures. Within such a theory, the inconsistencies presented above should vanish.

3.3 The approach of Sriboonchitta et al.

In a recent publication Sriboonchitta et al. (2010) develop a calculation scheme for the construction of a risk spectrum φ from a given utility function u. This method is a candidate for a relationship of the form (3). However, they do not present any nontrivial application.

Central to their method are two observations: First, the buying price $p_B(X)$ of a random future P&L X, can be defined according to expected utility theory as the real number, for which $\mathrm{E}_u(X - p_B) = 0$ holds. Furthermore, it can be estimated from a (hypothetical) random sample $\{x_i\}_{i=1,\ldots,N}$, by solving the equation $\frac{1}{N}\sum_{i=1}^N u(x_i - \hat{p}_B) = 0$. An equation of this form can be interpreted as M-estimator a_M for the translation parameter a of a distribution function[4]

[4]The index identifying the underlying random variable is suppressed from now on to simplify the notation.

$F_a(x) = F_0(x-a)$ with density $f_0(x-a)$, by defining f_0 in terms of

$$u(x) =: -\frac{d}{dx}\ln[f_0(x)], \tag{6}$$

see Sriboonchitta et al. (2010), and also Huber (1981). Second, an estimator for a spectral risk measure $\rho_\varphi(X)$ is given by $\hat{\rho}_\varphi(X) = \frac{1}{N}\sum_{i=1}^{N}\varphi(i/N)x_{(i)}$, with $\{x_{(i)}\}_{i=1,\dots,N}$ representing the sample in ascending order (Acerbi, 2002). Such an equation is equivalent to an L-estimator a_L for the translation parameter a of a distribution function $F_a(x) = F_0(x-a)$ with density $f_0(x-a)$ by defining F_0 in terms of

$$\varphi(F_0(x)) = -\frac{1}{A}\frac{d^2}{dx^2}\ln[f_0(x)], \tag{7}$$

where A normalizes φ (Sriboonchitta et al., 2010).

Given a utility function $u(x)$ with $u(0) = 0$, the method of Sriboonchitta et al. consists of the following steps:

1. Calculate an auxiliary density $f_0(x) = \exp\left[-\int_c^x u(t)dt\right]$, where c is used to normalize f_0.

2. Calculate the corresponding distribution function $F_0(x) = \int_{-\infty}^{x} f_0(t)dt$.

3. Calculate an auxiliary function $\Phi(p)$ solving $\Phi(F_0(x)) = -\frac{d^2}{dx^2}\ln[f_0(x)] = \frac{d}{dx}u(x)$, or equivalently $\Phi(p) = \frac{d}{dx}u(x)|_{x=F_0^{-1}(p)}$.

4. Calculate the norm $A := \int_0^1 \Phi(p)dp$ and the risk spectrum $\varphi(p) = \Phi(p)/A$.

This scheme ensures that the estimators a_L and a_M estimate the same entity, or equivalently that

$$\rho_\varphi(X) = -p_B \tag{8}$$

holds. This identification is the central hypothesis of Sriboonchitta et al.. Unfortunately, the validity of this hypothesis is verified only in the trivial case of a linear utility function $u(x) = x$, in which case $\varphi(p) = 1$ follows immediately. This leads to the fully consistent result $\rho_\varphi(X) = -\mathrm{E}[X] = -p_B$.

As a nontrivial example, consider the exponential utility function $u(x) = 1 - e^{-kx}$ with $k \geq 0$. This function was already contemplated by Sriboonchitta et al., but the calculations involved were considered too complex. Nevertheless they can be performed, as shown in the appendix, where the complete results for steps 1 to 4 of the calculation scheme are presented. For the auxiliary function $\Phi(p)$, cf. (28) and (29), one obtains

$$\Phi_k(p) = k\exp\left[-kF_0^{-1}(p)\right] \quad \text{with} \quad F_0(x) = k^2\Gamma\left[\frac{1}{k}\right]^{-1}\Gamma\left[\frac{1}{k}, \frac{1}{k}e^{-kx}\right], \tag{9}$$

where Γ is the incomplete Gamma-function $\Gamma[a,x] = \int_x^\infty t^{a-1}e^{-t}dt$ (Abramowitz and Stegun, 1972, p. 260). With the norm $A = k$, cf. (30), the risk spectrum

$$\varphi_k(p) = \exp\left[-kF_0^{-1}(p)\right]$$

follows, which finally defines the corresponding spectral risk measure

$$\rho_{\varphi_k}(X) = -\int_0^1 \exp\left[-kF_0^{-1}(p)\right] q_X(p)dp. \tag{10}$$

Again, $q_X(p)$ represents the quantile function of the P&L-distribution of the financial position under consideration.

The buying price $p_B(X)$ according to the utility function $u(x) = 1 - e^{-kx}$ is calculated from

$$\mathrm{E}_u[X - p_B] = \int_0^1 u(q_X(p) - p_B)dp = 1 - e^{kp_B}\int_0^1 e^{-kq_X(p)}dp = 0, \tag{11}$$

yielding

$$p_B(X) = -\frac{1}{k}\ln\left[\int_0^1 e^{-kq_X(p)}dp\right]. \tag{12}$$

Thus, the central hypothesis of Sriboonchitta et al. reads

$$\int_0^1 \exp\left[-kF_0^{-1}(p)\right] q_X(p)dp = \frac{1}{k}\ln\left[\int_0^1 e^{-kq_X(p)}dp\right]$$

in case of the exponential utility function $u(x) = 1 - e^{-kx}$.

In order to explore the implications of this result, consider the special case $k = 1$ and X being uniformly distributed on $[a, b]$. Equation (9) reduces to $F_0(x) = \exp[-\exp[-x]]$ (Gumbel-distribution) and thus $F_0^{-1}(p) = -\ln[-\ln[p]]$, yielding

$$\varphi_{k=1}(p) = -\ln[p].$$

The corresponding spectral risk measure $\rho_{\varphi_{k=1}}(X) = \int_0^1 \ln[p]q_X(p)dp$ can be easily evaluated using the quantile function $q_X(p) = (b-a)p + a$,

$$\rho_{\varphi_{k=1}}(X) = \int_0^1 \ln[p]\big((b-a)p + a\big)dp = -\frac{1}{4}(b + 3a). \tag{13}$$

For the buying price $p_B(X)$, one obtains

$$p_B(X) = -\ln\left[\int_0^1 e^{-(b-a)p-a}dp\right] = -\ln\left[\frac{e^{-a} - e^{-b}}{b-a}\right]. \tag{14}$$

In figure 2, $\rho_{\varphi_{k=1}}$ and $p_B(X)$ are displayed as function of the upper bound b of the uniform distribution. Obviously, equation 8 does not hold.

The inconsistencies revealed so far motivate a closer investigation of relation (8). Still consider $u(x) = 1 - e^{-kx}$, with arbitrary $k > 0$, $X_1 \sim U(a, b)$ and $X_2 \sim N(\mu, \sigma)$. The buying price $p_B(X_1)$ is a more general case of (14)

$$p_B(X_1) = -\ln\left[\frac{e^{-ka} - e^{-kb}}{b-a}\right]. \tag{15}$$

The buying price $p_B(X_2)$ can be calculated according to (11) by solving

$$\begin{aligned}\mathrm{E}_u[X_2 - p_B] &= \int_{-\infty}^{\infty} \frac{1 - e^{-k(x-p_B)}}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= 1 - \exp\left[-k(\mu - p_B) + \frac{k^2\sigma^2}{2}\right] = 0,\end{aligned}$$

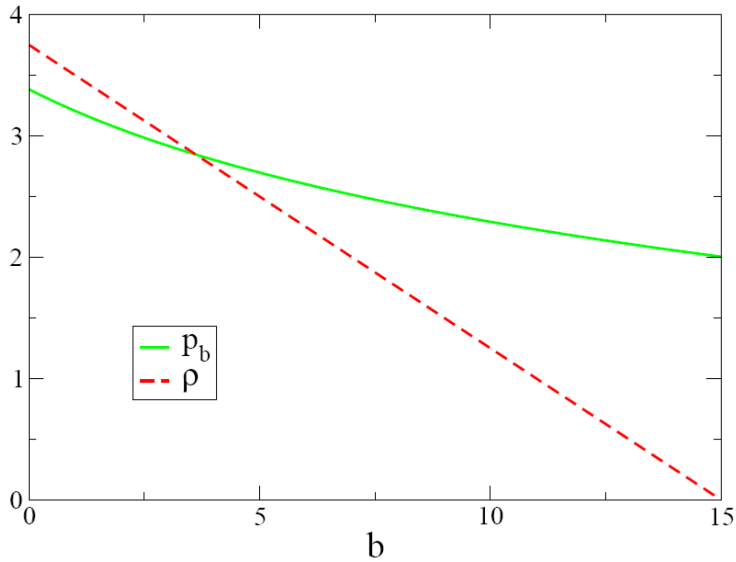

Figure 2: Buying price $p_B(X)$ (solid) and spectral risk measure $\rho_{\varphi_{k=1}}(X)$ (dashed) for an exponential utility function $u(x) = 1 - e^{-x}$ and a uniform distribution X on $[-5, b]$ as function of b.

which yields

$$p_B(X_2) = \mu - \frac{k}{2}\sigma^2. \tag{16}$$

In order for hypothesis (8) to be correct, $-p_B(X_1)$ and $-p_B(X_2)$ have to be spectral risk measures, i.e. each has to obey the axioms A1 to A6 of section 2. However, $-p_B(X_1)$ obviously violates axiom A2 and $-p_B(X_2)$ violates axiom A1. It follows that neither $-p_B(X_1)$ nor $-p_B(X_2)$ defines a spectral risk measure (in fact, both are not even coherent) and thus hypothesis (8) is violated.

Notice that similar results are obtained when the corresponding calculations are performed for a power-law utility function $u(x) = x^\alpha$ with $\alpha \in (0, 1)$ and $x \geq 0$, also briefly mentioned in Sriboonchitta et al. (2010).

In summary, for nontrivial applications the calculation scheme of Sriboonchitta et al. generates inconsistent results. Moreover, it was shown that the fundamental hypothesis (8) does not hold in general. Therefore, this method also fails in establishing a relationship between expected utility and spectral risk measures.

4 Distortion risk measures and decision theory

The overall result of the last section was a negative one: No consistent relationship between spectral risk measures and decision theory has yet been identified. In particular, the existing approaches to combine expected utility theory and spectral risk measures were proven to be inadequate in general. This serves as motivation to define risk measures from the somewhat different perspective of the premium principle, originated in the actuarial literature. The premium principle is introduced in subsection 4.1 and its application to expected utility theory is shown to support the results of section 3. In subsection 4.2 the dual theory of choice is introduced and contrasted with expected utility theory. Eventually, the application of the premium principle to the dual theory of choice serves as vehicle for the consistent modeling of risk aversion with spectral risk measures in subsection 4.3.

4.1 The Premium Principle

The starting point of the discussion is some arbitrary representation U of a preference relation according to (2) and the question, which premium P (risk compensation) has to be received by an insurer, in order to make a specific risk (encoded in the P&L-distribution) acceptable to him. The premium principle states that the premium to be received has to obey an indifference relation of the kind

$$U(X+P) = U(0). \tag{17}$$

The risk measure $\rho(X)$ obtained by defining $\rho(X) := P(X)$ by definition fulfills the relation

$$\rho(X) \leq \rho(Y) \Leftrightarrow U(X) \geq U(Y) \quad \text{for all} \quad X, Y \in L^0(\Omega, \mathcal{F}, P).$$

As a first example, consider expected utility theory and its representation $U(X) = \mathrm{E}_u[X]$ with utility function u, cf. (4). Within this representation, the premium principle (17) reads $U(X + P(X)) = \mathrm{E}_u[X + P(X)] = 0$. This so-called "zero-utility premium principle" (cf. Bühlmann, 2005) introduces a risk measure, which is not coherent (Tsanakas and Desli, 2003). A comparison with the definition of the buying price $\mathrm{E}_u[X - p_B] = 0$ of section 4 yields $p_B(X) = -P(X)$, and thus $\rho(X) = -p_B(X)$, which is identical to the hypothesis of Sriboonchitta et al., equation (8). This confirms the results of subsection 3.3.

4.2 Dual theory of choice

Motivated by empirical and theoretical criticism on the independence-axiom IA of section 3.1 (Puppe, 1991; Yaari, 1987), alternative decision theories have been derived by modifying this axiom (Puppe, 1991). Of particular importance for the matter at hand is the dual theory of choice, which replaces the original independence-axiom by the requirement

DA For any $X, Y, Z \in L^0(\Omega, \mathcal{F}, P)$, $x \in \mathbb{R}$ and $\lambda \in [0, 1]$ holds:
$X \succcurlyeq Y \Rightarrow \big(\lambda F_X * (1-\lambda) F_Z\big)(x) \succcurlyeq \big(\lambda F_Y * (1-\lambda) F_Z\big)(x)$.

The symbol $*$ indicates convex convolution, i.e. the terms in brackets are the distributions of the random variables $\lambda X + (1-\lambda)Z$ and $\lambda Y + (1-\lambda)Z$, respectively. Thus, not the independence of the convex combinations of the distributions, but the independence of the convex combinations of the random variables themselves is postulated.

In the framework of the dual theory of choice, preference relations have a representation

$$U(X) = \int_{-\infty}^{\infty} x d(h \circ F_X)(x) =: \mathrm{E}_h[X] \tag{18}$$

with some isotonic distortion function $h : [0,1] \to [0,1]$, obeying $h(0) = 0$ and $h(1) = 1$ (Yaari, 1987; Tsanakas and Desli, 2003). Unlike expected utility theory, which focuses on the expected value of a transformed future P&L of some position, the dual theory of choice considers the expected value of the proper P&L with respect to a transformed distribution, which corresponds to a change of the probability measure.

Second-order stochastic dominance or equivalently risk aversion of the decision maker (subsection 2.2) is guaranteed by requiring $\frac{d^2h}{dp^2} < 0$ (concave h, cf. Yaari, 1987). Furthermore, an analog to the Pratt-Arrow-coefficient can be defined by

$$r_{\mathrm{DPA}}(p) = -\frac{d^2h}{dp^2} \Big/ \frac{dh}{dp} \tag{19}$$

(Yaari, 1986). As mentioned before, we use a different sign convention and therefore the original results translate from convex to concave and from positive to negative.

4.3 Modeling of risk aversion with spectral risk measures

Now consider the representation $U(X) = \mathrm{E}_h[X]$ of the dual theory of choice (18). By applying the premium principle $U(X + P(X)) = \mathrm{E}_h[X + P(X)] = 0$, one obtains the risk measure

$$\rho_h(X) := P(X) = -\mathrm{E}_h[X] = -\int_{-\infty}^{\infty} x d(h \circ F_X)(x),$$

cf. section 4.1, which is known as distortion risk measure in the actuarial literature, cf. (Denneberg, 1990; Wang, 1996; Denuit et al., 2006). Substituting $x = F_X^{-1}(p)$ yields

$$\rho_h(X) = -\int_0^1 q_X(p) dh(p) = -\int_0^1 \frac{dh}{dp} q_X(p) dp, \tag{20}$$

which shows that $\rho_h(X)$ has the structure of a spectral risk measure. It can indeed be shown that for a concave distortion function h the risk measure $\rho_h(X)$ fulfills the axioms A1-A6 (Tsanakas and Desli, 2003, pp. 18). Thus, by the definition

$$\varphi_h(p) = \frac{d}{dp} h(p) \tag{21}$$

each concave distortion function induces a spectral risk measure $\rho_{\varphi_h}(X)$. Equally, any risk spectrum $\varphi(p)$ defines a concave distortion function $h(p)$ as integral

$$h(p) = \int_0^p \varphi(p) dp \tag{22}$$

of the differential equation (21). Thus, spectral risk measures are by the correspondences (21) and (22) equivalent to distortion measures with concave distortion functions.

This equivalency allows for the following conclusion, which constitutes the central result of this paper: Spectral risk measures can be derived from the premium principle using the representation $U(X) = \mathrm{E}_h[X]$ of the dual theory of choice. Thus, for any spectral risk measure $\rho_\varphi(X)$ the relation (3), i. e.

$$\rho_\varphi(X) \le \rho_\varphi(Y) \Leftrightarrow \mathrm{E}_h[X] \ge \mathrm{E}_h[Y] \quad \text{for any} \quad X, Y \in L^0(\Omega, \mathcal{F}, P),$$

naturally holds for $h(p)$ given by (22). Thus, any spectral risk measure uniquely reflects some preference relation, encoded in $h(p)$, which is consistent with the dual theory of choice. Furthermore as any risk spectrum corresponds to a concave distortion function, cf. equation (22), spectral risk measures reflect risk aversion of the decision maker, cf. 4.2. Coming from the other direction, any risk averse behavior of a decision maker, whose preference relation is consistent with the dual theory of choice, can be encoded in a risk spectrum, and eventually be modelled by a spectral risk measure.

The correspondence (21) between distortion function $h(p)$ and risk spectrum $\varphi(p)$ allows for the definition of a local measure for the risk aversion encoded in a given risk spectrum. Putting (21) into the modified Pratt-Arrow-coefficient r_{DPA}, equation (19), gives

$$r_{\mathrm{DPA}}(p) = -\frac{d^2h}{dp^2} / \frac{dh}{dp} = -\frac{d\varphi_h}{dp} / \varphi_h(p) \quad .$$

Thus by defining

$$r_{\mathrm{SPA}}(p) := -\frac{d\varphi}{dp} / \varphi(p) \tag{23}$$

a local measure of risk aversion ('spectral Pratt-Arrow' [SPA]) in terms of the risk spectrum φ in the integral representation of the spectral risk measure is defined. As illustration, consider a small $\Delta p > 0$ and expand the relative change of the risk to linear order:

$$\varphi_h(p - \Delta p)/\varphi_h(p) \approx 1 - \frac{1}{\varphi_h(p)} \frac{d\varphi_h}{dp} \Delta p = 1 + r_{\mathrm{SPA}}(p) \Delta p. \tag{24}$$

The coefficient $r_{\mathrm{SPA}}(p)$ thus measures the relative increase in weight of the quantiles in the direction of higher losses. Equation (24) can also be understood as differential equation to construct a risk spectrum $\varphi(p)$ from a given degree of risk aversion $r_{\mathrm{SPA}}(p)$.

The subsection concludes with four examples referring to former discussions.

Example 1: The constant risk spectrum $\varphi_h(p) = 1$ corresponds to the linear distortion function $h = p$ and has $r_{\mathrm{DPA}}(p) = r_{\mathrm{SPA}}(p) = 0$. This implies risk-neutral behavior of the decision maker. Note that this also implies that the expected shortfall, which is the spectral risk measure defined by the risk spectrum $\varphi_h(p) = \frac{1}{\alpha}\Theta(\alpha - p)$ reflects risk neutral behavior inside the range of relevant quantiles (Acerbi, 2004).

Example 2: According to appendix A, the exponential risk spectrum $\bar{\varphi}_k^{(1)}(p)$ of subsection 3.2 corresponds to the risk spectrum $\varphi_k^{(1)}(p) = \frac{ke^{-kp}}{1-e^{-k}}$ within the convention of losses described by negative numbers, adopted here. The distortion function follows from (22) as $h(p) = \frac{k}{1-e^{-k}} \int_0^q e^{-kt} dt = \frac{1-e^{-kp}}{1-e^{-k}}$. The

Pratt-Arrow-coefficient is $r_{\mathrm{DPA}}(p) = k$, thus reflecting a constant absolute risk aversion.

Example 3: The risk spectrum $\bar{\varphi}^{(2)}_{\gamma}(p)$, with $0 < \gamma < 1$, introduced in subsection 3.2, corresponds to the risk spectrum $\varphi^{(2)}_{\gamma} = \gamma p^{\gamma-1}$, cf. appendix A. It has the property to induce a spectral risk measure, which falls with increasing parameter γ, see figure 1. This behavior becomes evident by calculating the corresponding Pratt-Arrow-coefficient $r_{\mathrm{SPA}}(p) = \frac{1-\gamma}{p} > 0$, which falls with increasing γ, showing that increasing γ indeed corresponds to a less risk averse behavior.

One arrives at the same conclusion using the corresponding distortion function

$$h(p) = \int_0^p \varphi^{(2)}_{\gamma}(t)dt = p^{\gamma}.$$

For γ increasing from close to 0 to 1, $h(p)$ becomes less and less concave until reaching risk neutral behavior at $\gamma = 1$. The Pratt-Arrow-coefficient $r_{\mathrm{DPA}}(p) = \frac{1-\gamma}{p} > 0$ illustrates this.

5 Conclusions

The relation between expected utility theory, the dual theory of choice, and spectral risk measures has thoroughly been investigated. In particular, an answer to the question of how to model subjective risk aversion with spectral risk measures was provided. This question is far from being trivial, as was revealed by the analysis of recent scientific approaches to this problem. It was demonstrated that these approaches, albeit most intriguing ideas are involved, fail to consistently model risk averse behavior of a decision maker, whose preference relation is consistent with expected utility theory. The key problem is the use of expected utility theory.

Using the premium principle and the result from actuarial literature, that there is an intimate relationship between spectral risk measures and the so called distortion risk measures, it was shown, that spectral risk measures can be used to model any risk averse behavior of a decision maker, whose preference relation is consistent with the dual theory of choice. Building upon this result, a local measure for the risk aversion encoded in the risk spectrum of the spectral risk measure was defined. These findings were illustrated on multiple examples.

A Sign conventions for measuring losses

Throughout the financial literature two sign conventions for measuring losses are equally being used: either losses are described as negative numbers, implying that, say, an outcome of −100$ means that a loss of 100$ is realized or losses are described as positive numbers, implying that an outcome of 100$ means that a loss of 100$ is realized. In this appendix, the transformation from one convention to the other in the context of spectral risk measures is presented to enhance the comprehensibility of the discussions in the main text.

First, consider a financial position with random P&L-variable X and assume that losses are described by negative numbers. Consider some spectral risk measure

$$\rho_\varphi(X) = -\int_0^1 \varphi(p) q_X(p) dp$$

as defined in equation (1). Now require that by changing the sign of the losses, i. e. by setting $X \curvearrowright -X$, the risk assigned to the P&L-variable does not change. Formally, an equivalent risk spectrum $\bar{\varphi}(x)$ has to be constructed, such that

$$\rho_{\bar{\varphi}}(-X) = -\rho_\varphi(X) \tag{25}$$

holds. Using the equality $q_X(p) = -q_{-X}(1-p)$ for the quantiles, the calculation is straightforward:

$$\begin{aligned}\rho_\varphi(X) &= -\int_0^1 \varphi(p) q_X(p) dp = \int_0^1 \varphi(p) q_{-X}(1-p) dp \\ &= \int_0^1 \varphi(1-\tilde{p}) q_{-X}(\tilde{p}) d\tilde{p}\end{aligned}$$

with the substitution $\tilde{p} = 1 - p$ being made from first to second line. Thus, by choosing

$$\bar{\varphi}(p) := \varphi(1-p) \tag{26}$$

the required equality (25) holds.

As examples, consider the three risk spectrums $\bar{\varphi}_k^{(1)}, \bar{\varphi}_\gamma^{(2)}$ and $\bar{\varphi}_\gamma^{(3)}$ introduced by Dowd et al. with the convention that losses are described by positive numbers, cf. section 3.2. Using equation (26) one can write down the equivalent risk spectrums $\varphi_k^{(1)}, \varphi_\gamma^{(2)}$ and $\varphi_\gamma^{(3)}$ with the convention that losses are described by negative numbers immediately:

$$\begin{aligned}\varphi_k^{(1)} &= \frac{k e^{-kp}}{1 - e^{-k}} \\ \varphi_\gamma^{(2)} &= \gamma p^{\gamma - 1} \\ \varphi_\gamma^{(3)} &= \gamma (1-p)^{\gamma-1} \quad .\end{aligned}$$

In section 4.3, the relationship between risk spectrum $\varphi(p)$ and concave distortion function

$$\varphi_h(p) = \frac{d}{dp} h(p)$$

was derived within the convention of describing losses by negative numbers. Switching to the convention of describing losses by positive numbers, an equivalent relation

$$\bar{\varphi}_{\bar{h}}(p) = \frac{d}{dp} \bar{h}(p)$$

has to hold. In order to calculate the relation between h and $\bar{h}$, integrate the equation above and use equation (26):

$$\begin{aligned}\bar{h}(p) &= \int_0^p \bar{\varphi}(t)dt = \int_0^p \varphi(1-t)dt \\ &= -\int_1^{1-p} \varphi(y)dy = \int_0^1 \varphi(y)dy - \int_0^{1-p} \varphi(y)dy \\ &= 1 - h(1-p) \quad ,\end{aligned}$$

where the substitution $y = 1 - t$, equation (22), and the fact that $\varphi(p)$ is normalized was used. This function $\bar{h}$ is known as conjugated distortion function and is convex, if h is concave. This consistently reflects risk aversion within the convention of decribing losses by positive numbers, cf. (Yaari, 1987; Denuit et al., 2006). Furthermore

$$\bar{\varphi}(p) = \frac{d\bar{h}(p)}{dp} = \frac{dh}{dx}|_{x=1-p} = \varphi(1-p)$$

holds. Within the convention of describing losses by negative numbers, the distortion risk measure

$$\rho_h(X) = -\int_0^1 q_X(p)dh(p) = -\int_0^1 \frac{dh}{dp} q_X(p)dp$$

was defined, cf. equation (20). Using the relations derived above, the transformation to the convention of describing losses by positive numbers reads

$$\begin{aligned}\rho_h(X) &= -\int_0^1 q_X(p)dh(p) = -\int_0^1 \frac{dh}{dp} q_X(p)dp \\ &= \int_0^1 \frac{dh}{dp} q_{-X}(1-p)dp \\ &= \int_0^1 \frac{dh}{dx}|_{x=1-\tilde{p}} q_{-X}(\tilde{p})d\tilde{p} \\ &= \int_0^1 \frac{d\bar{h}(\tilde{p})}{d\tilde{p}} q_{-X}(\tilde{p})d\tilde{p} \\ &= \int_0^1 q_{-X}(\tilde{p})d\bar{h}(\tilde{p}) \\ &= -\rho_{\bar{h}}(-X),\end{aligned}$$

where the substitution $\tilde{p} = 1 - p$ was used. The transformed result agrees with equation (3.8) and (3.9) of (Denuit et al., 2006).

B Exponential utility function

In this appendix, we represent the calculation of the exponential utility function, used as example in the discussion of the method of Sriboonchitta et al. (2010) in section 4.

Consider the utility function

$$u(x) = 1 - e^{-kx} \quad \text{with} \quad k > 0. \tag{27}$$

Calculate the auxiliary function $f_0(x) = \exp\left[-\int_c^x u(t)dt\right]$:

$$\int_c^x u(t)dt = \int_c^x (1 - \exp[-kt])dt = x + \frac{1}{k}\exp[-kx] + \tilde{c},$$

and thus

$$f_0(x) = N \exp\left[-\left(\frac{1}{k}\exp[-kx] + x\right)\right].$$

The norm N will be determined below. Calculate the second auxiliary function $F_0(x) = \int_{-\infty}^x f_0(t)dt$:

$$F_0(x) = N \int_{-\infty}^x \exp\left[-\left(\frac{1}{k}\exp[-kt] + t\right)\right] dt.$$

Substituting $z := \exp[-kt]$ yields

$$F_0(x) = \frac{N}{k} \int_{\exp[-kx]}^{\infty} \exp\left[-\frac{z}{k}\right] z^{1/k-1} dz.$$

Using equation (3.381.3) of Gradshteyn and Ryzhik (1980) with $u = \exp[-kx]$, $\mu = 1/k$ and $\nu = 1/k$ we arrive at

$$F_0(x) = \frac{N}{k}\mu^{-\nu}\Gamma[\nu, \mu u] = N(1/k)^{-1/k+1}\Gamma\left[\frac{1}{k}, \frac{1}{k}\exp[-kx]\right],$$

where Γ represents the incomplete Gamma function $\Gamma[a,x] = \int_x^\infty e^{-t}t^{a-1}dt$ (Abramowitz and Stegun, 1972, p. 260).

The norm N can be calculated from the condition

$$\int_{-\infty}^{\infty} f_0(t)dt = F_0(\infty) = 1,$$

yielding

$$N^{-1} = (1/k)^{-1/k+1}\Gamma\left[\frac{1}{k}, 0\right] = (1/k)^{-1/k+1}\Gamma\left[\frac{1}{k}\right].$$

Eventually, the second auxiliary function is obtained

$$F_0(x) = k^2\Gamma\left[\frac{1}{k}\right]^{-1}\Gamma\left[\frac{1}{k}, \frac{1}{k}\exp[-kx]\right]. \tag{28}$$

Calculate the third auxiliary function $\Phi(F_0(x)) = \frac{d}{dx}u(x)$, respectively $\Phi(p) = \frac{d}{dx}u\big|_{x=F_0^{-1}(p)}$:

$$\Phi(p) = k\exp[-kF_0^{-1}(p)] = k\exp[-kq_X(p)]. \tag{29}$$

The quantile function $q_X(p) = F_0^{-1}(p)$ cannot be expressed in a closed form. Finally we compute the integral

$$A := \int_0^1 \Phi(p)dp = k \int_0^1 \exp\bigl[-kF_0^{-1}(p)\bigr]dp.$$

Substituting $x := F_0^{-1}(t)$ yields

$$A = k \int_{-\infty}^{\infty} \frac{dF_0}{dx} \exp[-kx]dx = k \int_{-\infty}^{\infty} f_0(x) \exp[-kx]dx.$$

Using the density, one obtains

$$A = (1/k)^{1/k-1} \Gamma \left[\frac{1}{k}\right]^{-1} k \int_{-\infty}^{\infty} \exp\left[-\left(\frac{1}{k}\exp[-kx] + x\right)\right] \exp[-kx]dx,$$

which after substituting $z := \exp[-kx]$ yields

$$A = (1/k)^{1/k-1} \Gamma \left[\frac{1}{k}\right]^{-1} \int_0^{\infty} \exp\left[-\frac{z}{k}\right] z^{1/k} dz.$$

Equation (3.381.4) in Gradshteyn and Ryzhik (1980) with $\mu = 1/k$ and $\nu = 1/k + 1$ leads to

$$A = (1/k)^{1/k-1} \Gamma \left[\frac{1}{k}\right]^{-1} (1/k)^{-1/k-1} \Gamma \left[\frac{1}{k} + 1\right] = k, \tag{30}$$

where the recursion formula $\Gamma[z+1] = z\Gamma[z]$ was used (cf. Abramowitz and Stegun, 1972, p. 256). Eventually, the risk spectrum

$$\varphi_k(p) = \Phi(p)/A = \exp\bigl[-kF_0^{-1}(p)\bigr] \tag{31}$$

follows.

References

Abramowitz, Milton and Irene A Stegun (1972): *Handbook of mathematical functions.* Dover Publications, New York, 9th edn.

Acerbi, Carlo (2002): Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking & Finance*, 26:1505.

Acerbi, Carlo (2004): Coherent Representations of Subjective Risk-Aversion. In: *Risk Measures for the 21st Century*, ed. Giorgio Szegö, John Wiley & Sons, Chichester, p. 147.

Albanese, Claudio and Stephan Lawi (2004): Spectral Risk Measures for Credit Portfolios. In: *Risk Measures for the 21st Century*, ed. Giorgio Szegö, John Wiley & Sons, Chichester, p. 209.

Arrow, K J (1971): *Essays in the Theory of Risk-Bearing.* Markham Publishing Company, Chicago, 1st edn.

Artzner, Philippe; Freddy Delbaen; Jean-Marc Eber; and David Heath (1999): Coherent Measures of Risk. *Mathematical Finance*, 9:203.

Bawa, Vijay S (1975): Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, 2:95.

Bernoulli, D (1738): Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae [translated by Sommer, L. in Econometrica 22, 23 (1954)]*, 5:175.

Bühlmann, Hans (2005): *Mathematical methods in risk theory*. Springer Verlag, New York, 2nd edn.

Denneberg, D (1990): Distorted probabilities and insurance premiums. *Methods of Operations Research*, 63:3.

Denuit, Michel; Jan Dhaene; Marc Goovaerts; Rob Kaas; and Roger Laeven (2006): Risk measurement with equivalent utility principles. *Statistics & Decisions*, 24:1.

Dowd, Kevin; John Cotter; and Ghulam Sorwar (2008): Spectral Risk Measures: Properties and Limitations. *Journal of Financial Services Research*, 34:61.

Föllmer, Hans and Alexander Schied (2002): *Stochastic Finance*. Walter de Gruyter, Berlin, 1st edn.

Galassi, M; B Gough; and G Jungman (2009): *GNU Scientific Library Reference Manual*. Network Theory Limited.

Gradshteyn, Izrail Solomonovich and Iosif Moiseevich Ryzhik (1980): *Table of integrals, series and products*. Academic Press, Orlando, 3rd edn.

Huber, Peter J (1981): *Robust Statistics*. John Wiley & Sons, Chichester, 1st edn.

Keeney, Ralph L and Howard Raiffa (1993): *Decisions with multiple objectives*. Cambridge University Press, Cambridge, 1st edn.

Knuth, D E (2009): *The art of computer programming: Fascicles 0-4*. Addison Wesley, London, 1st edn.

Markowitz, Harry (1952): Portfolio Selection. *Journal of Finance*, 7:77.

Markowitz, Harry (1959): *Portfolio Selection*. John Wiley & Sons, Chichester, 1st edn.

Matsumoto, M and T Nishimura (1998): Mersenne Twister: A 623-dimensionally equidistributed uniform pseudorandom number generator. *ACM Transactions on Modeling and Computer Simulation*, 8:3.

Pratt, J W (1964): Risk aversion in the small and in the large. *Econometrica*, 32:122.

Puppe, Clemens (1991): *Distorted probabilities and choice under risk*. Springer Verlag, Berlin, 1st edn.

Raiffa, Howard (1970): *Decision Analysis*. Addison-Wesley, New York, 2nd edn.

Rothschild, M and J Stiglitz (1970): Increasing risk I: A definition. *Journal of Econometric Theory*, 2:225.

Sriboonchitta, S; Hung T Nguyen; and V Kreinovich (2010): How to relate spectral risk measures and utilities. *International Journal of Intelligent Technologies and Applied Statistics*, 3:141.

Tao, M; X Ren; and Y Yang (2009): Construction and choice of risk spectrum function. *Journal of Beijing University of Chemical Technology (Natural Science Edition)*, 36:105.

Tasche, Dirk (2002): Expected shortfall and beyond. *Journal of Banking & Finance*, 26:1519.

Tsanakas, A. and E. Desli (2003): Risk measures and theories of choice. *British Actuarial Journal*, 9:959.

von Neumann, John and O Morgenstern (1947): *Theory of games and economic behavior*. Princeton University Press, Princeton, 2nd edn.

Wang, Shaun (1996): Premium calculation by transforming the layer premium density. *Astin Bulletin*, 26:71.

Yaari, M E (1986): Univariate and multivariate comparisons of risk aversion. In: *Uncertainty, Information, and Communication*, eds. P H Walter; M S Ross; and D A Starrett, Cambridge University Press, Cambridge. 1st edn.

Yaari, Menahem E (1987): The dual theory of choice under risk. *Econometrica*, 55:95.

Die Diskussionspapiere ab Nr. 183 (1992) bis heute, können Sie im Internet unter http://www.fernuni-hagen.de/FBWIWI/ einsehen und zum Teil downloaden.
Die **Titel** der Diskussionspapiere von Nr 1 (1975) bis 182 (1991) können bei Bedarf in der Fakultät für Wirtschaftswissenschaft angefordert werden:
FernUniversität, z. Hd. Frau Huber oder Frau Mette, Postfach 940, 58084 Hagen .
Die Diskussionspapiere selber erhalten Sie nur in den Bibliotheken.

Nr	**Jahr**	**Titel**	**Autor/en**
322	2001	Spreading Currency Crises: The Role of Economic Interdependence	Berger, Wolfram Wagner, Helmut
323	2002	Planung des Fahrzeugumschlags in einem Seehafen-Automobilterminal mittels eines Multi-Agenten-Systems	Fischer, Torsten Gehring, Hermann
324	2002	A parallel tabu search algorithm for solving the container loading problem	Bortfeldt, Andreas Gehring, Hermann Mack, Daniel
325	2002	Die Wahrheit entscheidungstheoretischer Maximen zur Lösung von Individualkonflikten - Unsicherheitssituationen -	Mus, Gerold
326	2002	Zur Abbildungsgenauigkeit des Gini-Koeffizienten bei relativer wirtschaftlicher Konzentration	Steinrücke, Martin
327	2002	Entscheidungsunterstützung bilateraler Verhandlungen über Auftragsproduktionen - eine Analyse aus Anbietersicht	Steinrücke, Martin
328	2002	Die Relevanz von Marktzinssätzen für die Investitionsbeurteilung – zugleich eine Einordnung der Diskussion um die Marktzinsmethode	Terstege, Udo
329	2002	Evaluating representatives, parliament-like, and cabinet-like representative bodies with application to German parliament elections 2002	Tangian, Andranik S.
330	2002	Konzernabschluss und Ausschüttungsregelung im Konzern. Ein Beitrag zur Frage der Eignung des Konzernabschlusses als Ausschüttungsbemessungsinstrument	Hinz, Michael
331	2002	Theoretische Grundlagen der Gründungsfinanzierung	Bitz, Michael
332	2003	Historical background of the mathematical theory of democracy	Tangian, Andranik S.
333	2003	MCDM-applications of the mathematical theory of democracy: choosing travel destinations, preventing traffic jams, and predicting stock exchange trends	Tangian, Andranik S.

334	2003	Sprachregelungen für Kundenkontaktmitarbeiter – Möglichkeiten und Grenzen	Fließ, Sabine Möller, Sabine Momma, Sabine Beate
335	2003	A Non-cooperative Foundation of Core-Stability in Positive Externality NTU-Coalition Games	Finus, Michael Rundshagen, Bianca
336	2003	Combinatorial and Probabilistic Investigation of Arrow's dictator	Tangian, Andranik
337	2003	A Grouping Genetic Algorithm for the Pickup and Delivery Problem with Time Windows	Pankratz, Giselher
338	2003	Planen, Lernen, Optimieren: Beiträge zu Logistik und E-Learning. Festschrift zum 60 Geburtstag von Hermann Gehring	Bortfeldt, Andreas Fischer, Torsten Homberger, Jörg Pankratz, Giselher Strangmeier, Reinhard
339a	2003	Erinnerung und Abruf aus dem Gedächtnis Ein informationstheoretisches Modell kognitiver Prozesse	Rödder, Wilhelm Kuhlmann, Friedhelm
339b	2003	Zweck und Inhalt des Jahresabschlusses nach HGB, IAS/IFRS und US-GAAP	Hinz, Michael
340	2003	Voraussetzungen, Alternativen und Interpretationen einer zielkonformen Transformation von Periodenerfolgsrechnungen – ein Diskussionsbeitrag zum LÜCKE-Theorem	Terstege, Udo
341	2003	Equalizing regional unemployment indices in West and East Germany	Tangian, Andranik
342	2003	Coalition Formation in a Global Warming Game: How the Design of Protocols Affects the Success of Environmental Treaty-Making	Eyckmans, Johan Finus, Michael
343	2003	Stability of Climate Coalitions in a Cartel Formation Game	Finus, Michael van Ierland, Ekko Dellink, Rob
344	2003	The Effect of Membership Rules and Voting Schemes on the Success of International Climate Agreements	Finus, Michael J.-C., Altamirano-Cabrera van Ierland, Ekko
345	2003	Equalizing structural disproportions between East and West German labour market regions	Tangian, Andranik
346	2003	Auf dem Prüfstand: Die geldpolitische Strategie der EZB	Kißmer, Friedrich Wagner, Helmut

347	2003	Globalization and Financial Instability: Challenges for Exchange Rate and Monetary Policy	Wagner, Helmut
348	2003	Anreizsystem Frauenförderung – Informationssystem Gleichstellung am Fachbereich Wirtschaftswissenschaft der FernUniversität in Hagen	Fließ, Sabine Nonnenmacher, Dirk
349	2003	Legitimation und Controller	Pietsch, Gotthard Scherm, Ewald
350	2003	Controlling im Stadtmarketing – Ergebnisse einer Primärerhebung zum Hagener Schaufenster-Wettbewerb	Fließ, Sabine Nonnenmacher, Dirk
351	2003	Zweiseitige kombinatorische Auktionen in elektronischen Transportmärkten – Potenziale und Probleme	Pankratz, Giselher
352	2003	Methodisierung und E-Learning	Strangmeier, Reinhard Bankwitz, Johannes
353 a	2003	A parallel hybrid local search algorithm for the container loading problem	Mack, Daniel Bortfeldt, Andreas Gehring, Hermann
353 b	2004	Übernahmeangebote und sonstige öffentliche Angebote zum Erwerb von Aktien – Ausgestaltungsmöglichkeiten und deren Beschränkung durch das Wertpapiererwerbs- und Übernahmegesetz	Wirtz, Harald
354	2004	Open Source, Netzeffekte und Standardisierung	Maaß, Christian Scherm, Ewald
355	2004	Modesty Pays: Sometimes!	Finus, Michael
356	2004	Nachhaltigkeit und Biodiversität	Endres, Alfred Bertram, Regina
357	2004	Eine Heuristik für das dreidimensionale Strip-Packing-Problem	Bortfeldt, Andreas Mack, Daniel
358	2004	Netzwerkökonomik	Martiensen, Jörn
359	2004	Competitive versus cooperative Federalism: Is a fiscal equalization scheme necessary from an allocative point of view?	Arnold, Volker
360	2004	Gefangenendilemma bei Übernahmeangeboten? Eine entscheidungs- und spieltheoretische Analyse unter Einbeziehung der verlängerten Annahmefrist gem. § 16 Abs. 2 WpÜG	Wirtz, Harald

361	2004	Dynamic Planning of Pickup and Delivery Operations by means of Genetic Algorithms	Pankratz, Giselher
362	2004	Möglichkeiten der Integration eines Zeitmanagements in das Blueprinting von Dienstleistungsprozessen	Fließ, Sabine Lasshof, Britta Meckel, Monika
363	2004	Controlling im Stadtmarketing - Eine Analyse des Hagener Schaufensterwettbewerbs 2003	Fließ, Sabine Wittko, Ole
364	2004	Ein Tabu Search-Verfahren zur Lösung des Timetabling-Problems an deutschen Grundschulen	Desef, Thorsten Bortfeldt, Andreas Gehring, Hermann
365	2004	Die Bedeutung von Informationen, Garantien und Reputation bei integrativer Leistungserstellung	Prechtl, Anja Völker-Albert, Jan-Hendrik
366	2004	The Influence of Control Systems on Innovation: An empirical Investigation	Littkemann, Jörn Derfuß, Klaus
367	2004	Permit Trading and Stability of International Climate Agreements	Altamirano-Cabrera, Juan-Carlos Finus, Michael
368	2004	Zeitdiskrete vs. zeitstetige Modellierung von Preismechanismen zur Regulierung von Angebots- und Nachfragemengen	Mazzoni, Thomas
369	2004	Marktversagen auf dem Softwaremarkt? Zur Förderung der quelloffenen Softwareentwicklung	Christian Maaß Ewald Scherm
370	2004	Die Konzentration der Forschung als Weg in die Sackgasse? Neo-Institutionalistische Überlegungen zu 10 Jahren Anreizsystemforschung in der deutschsprachigen Betriebswirtschaftslehre	Süß, Stefan Muth, Insa
371	2004	Economic Analysis of Cross-Border Legal Uncertainty: the Example of the European Union	Wagner, Helmut
372	2004	Pension Reforms in the New EU Member States	Wagner, Helmut
373	2005	Die Bundestrainer-Scorecard Zur Anwendbarkeit des Balanced Scorecard Konzepts in nicht-ökonomischen Fragestellungen	Eisenberg, David Schulte, Klaus

374	2005	Monetary Policy and Asset Prices: More Bad News for ‚Benign Neglect“	Berger, Wolfram Kißmer, Friedrich Wagner, Helmut
375	2005	Zeitstetige Modellbildung am Beispiel einer volkswirtschaftlichen Produktionsstruktur	Mazzoni, Thomas
376	2005	Economic Valuation of the Environment	Endres, Alfred
377	2005	Netzwerkökonomik – Eine vernachlässigte theoretische Perspektive in der Strategie-/Marketingforschung?	Maaß, Christian Scherm, Ewald
378	2005	Diversity management`s diffusion and design: a study of German DAX-companies and Top-50-U.S.-companies in Germany	Süß, Stefan Kleiner, Markus
379	2005	Fiscal Issues in the New EU Member Countries – Prospects and Challenges	Wagner, Helmut
380	2005	Mobile Learning – Modetrend oder wesentlicher Bestandteil lebenslangen Lernens?	Kuszpa, Maciej Scherm, Ewald
381	2005	Zur Berücksichtigung von Unsicherheit in der Theorie der Zentralbankpolitik	Wagner, Helmut
382	2006	Effort, Trade, and Unemployment	Altenburg, Lutz Brenken, Anke
383	2006	Do Abatement Quotas Lead to More Successful Climate Coalitions?	Altamirano-Cabrera, Juan-Carlos Finus, Michael Dellink, Rob
384	2006	Continuous-Discrete Unscented Kalman Filtering	Singer, Hermann
385	2006	Informationsbewertung im Spannungsfeld zwischen der Informationstheorie und der Betriebswirtschaftslehre	Reucher, Elmar
386	2006	The Rate Structure Pattern: An Analysis Pattern for the Flexible Parameterization of Charges, Fees and Prices	Pleß, Volker Pankratz, Giselher Bortfeldt, Andreas
387a	2006	On the Relevance of Technical Inefficiencies	Fandel, Günter Lorth, Michael
387b	2006	Open Source und Wettbewerbsstrategie - Theoretische Fundierung und Gestaltung	Maaß, Christian
388	2006	Induktives Lernen bei unvollständigen Daten unter Wahrung des Entropieprinzips	Rödder, Wilhelm

389	2006	Banken als Einrichtungen zur Risikotransformation	Bitz, Michael
390	2006	Kapitalerhöhungen börsennotierter Gesellschaften ohne börslichen Bezugsrechtshandel	Terstege, Udo Stark, Gunnar
391	2006	Generalized Gauss-Hermite Filtering	Singer, Hermann
392	2006	Das Göteborg Protokoll zur Bekämpfung grenzüberschreitender Luftschadstoffe in Europa: Eine ökonomische und spieltheoretische Evaluierung	Ansel, Wolfgang Finus, Michael
393	2006	Why do monetary policymakers lean with the wind during asset price booms?	Berger, Wolfram Kißmer, Friedrich
394	2006	On Supply Functions of Multi-product Firms with Linear Technologies	Steinrücke, Martin
395	2006	Ein Überblick zur Theorie der Produktionsplanung	Steinrücke, Martin
396	2006	Parallel greedy algorithms for packing unequal circles into a strip or a rectangle	Timo Kubach, Bortfeldt, Andreas Gehring, Hermann
397	2006	C&P Software for a cutting problem of a German wood panel manufacturer – a case study	Papke, Tracy Bortfeldt, Andreas Gehring, Hermann
398	2006	Nonlinear Continuous Time Modeling Approaches in Panel Research	Singer, Hermann
399	2006	Auftragsterminierung und Materialflussplanung bei Werkstattfertigung	Steinrücke, Martin
400	2006	Import-Penetration und der Kollaps der Phillips-Kurve	Mazzoni, Thomas
401	2006	Bayesian Estimation of Volatility with Moment-Based Nonlinear Stochastic Filters	Grothe, Oliver Singer, Hermann
402	2006	Generalized Gauss-H ermite Filtering for Multivariate Diffusion Processes	Singer, Hermann
403	2007	A Note on Nash Equilibrium in Soccer	Sonnabend, Hendrik Schlepütz, Volker
404	2007	Der Einfluss von Schaufenstern auf die Erwartungen der Konsumenten - eine explorative Studie	Fließ, Sabine Kudermann, Sarah Trell, Esther
405	2007	Die psychologische Beziehung zwischen Unternehmen und freien Mitarbeitern: Eine empirische Untersuchung des Commitments und der arbeitsbezogenen Erwartungen von IT-Freelancern	Süß, Stefan
406	2007	An Alternative Derivation of the Black-Scholes Formula	Zucker, Max Singer, Hermann

407	2007	Computational Aspects of Continuous-Discrete Extended Kalman-Filtering	Mazzoni, Thomas
408	2007	Web 2.0 als Mythos, Symbol und Erwartung	Maaß, Christian Pietsch, Gotthard
409	2007	„Beyond Balanced Growth“: Some Further Results	Stijepic, Denis Wagner, Helmut
410	2007	Herausforderungen der Globalisierung für die Entwicklungsländer: Unsicherheit und geldpolitisches Risikomanagement	Wagner, Helmut
411	2007	Graphical Analysis in the New Neoclassical Synthesis	Giese, Guido Wagner, Helmut
412	2007	Monetary Policy and Asset Prices: The Impact of Globalization on Monetary Policy Trade-Offs	Berger, Wolfram Kißmer, Friedrich Knütter, Rolf
413	2007	Entropiebasiertes Data Mining im Produktdesign	Rudolph, Sandra Rödder, Wilhelm
414	2007	Game Theoretic Research on the Design of International Environmental Agreements: Insights, Critical Remarks and Future Challenges	Finus, Michael
415	2007	Qualitätsmanagement in Unternehmenskooperationen - Steuerungsmöglichkeiten und Datenintegrationsprobleme	Meschke, Martina
416	2007	Modernisierung im Bund: Akteursanalyse hat Vorrang	Pietsch, Gotthard Jamin, Leander
417	2007	Inducing Technical Change by Standard Oriented Evirnonmental Policy: The Role of Information	Endres, Alfred Bertram, Regina Rundshagen, Bianca
418	2007	Der Einfluss des Kontextes auf die Phasen einer SAP-Systemimplementierung	Littkemann, Jörn Eisenberg, David Kuboth, Meike
419	2007	Endogenous in Uncertainty and optimal Monetary Policy	Giese, Guido Wagner, Helmut
420	2008	Stockkeeping and controlling under game theoretic aspects	Fandel, Günter Trockel, Jan
421	2008	On Overdissipation of Rents in Contests with Endogenous Intrinsic Motivation	Schlepütz, Volker
422	2008	Maximum Entropy Inference for Mixed Continuous-Discrete Variables	Singer, Hermann
423	2008	Eine Heuristik für das mehrdimensionale Bin Packing Problem	Mack, Daniel Bortfeldt, Andreas
424	2008	Expected A Posteriori Estimation in Financial Applications	Mazzoni, Thomas
425	2008	A Genetic Algorithm for the Two-Dimensional Knapsack Problem with Rectangular Pieces	Bortfeldt, Andreas Winter, Tobias
426	2008	A Tree Search Algorithm for Solving the Container Loading Problem	Fanslau, Tobias Bortfeldt, Andreas
427	2008	Dynamic Effects of Offshoring	Stijepic, Denis Wagner, Helmut

428	2008	Der Einfluss von Kostenabweichungen auf das Nash-Gleichgewicht in einem nicht-kooperativen Disponenten-Controller-Spiel	Fandel, Günter Trockel, Jan
429	2008	Fast Analytic Option Valuation with GARCH	Mazzoni, Thomas
430	2008	Conditional Gauss-Hermite Filtering with Application to Volatility Estimation	Singer, Hermann
431	2008	Web 2.0 auf dem Prüfstand: Zur Bewertung von Internet-Unternehmen	Christian Maaß Gotthard Pietsch
432	2008	Zentralbank-Kommunikation und Finanzstabilität – Eine Bestandsaufnahme	Knütter, Rolf Mohr, Benjamin
433	2008	Globalization and Asset Prices: Which Trade-Offs Do Central Banks Face in Small Open Economies?	Knütter, Rolf Wagner, Helmut
434	2008	International Policy Coordination and Simple Monetary Policy Rules	Berger, Wolfram Wagner, Helmut
435	2009	Matchingprozesse auf beruflichen Teilarbeitsmärkten	Stops, Michael Mazzoni, Thomas
436	2009	Wayfindingprozesse in Parksituationen - eine empirische Analyse	Fließ, Sabine Tetzner, Stefan
437	2009	ENTROPY-DRIVEN PORTFOLIO SELECTION a downside and upside risk framework	Rödder, Wilhelm Gartner, Ivan Ricardo Rudolph, Sandra
438	2009	Consulting Incentives in Contests	Schlepütz, Volker
439	2009	A Genetic Algorithm for a Bi-Objective Winner-Determination Problem in a Transportation-Procurement Auction"	Buer, Tobias Pankratz, Giselher
440	2009	Parallel greedy algorithms for packing unequal spheres into a cuboidal strip or a cuboid	Kubach, Timo Bortfeldt, Andreas Tilli, Thomas Gehring, Hermann
441	2009	SEM modeling with singular moment matrices Part I: ML-Estimation of time series	Singer, Hermann
442	2009	SEM modeling with singular moment matrices Part II: ML-Estimation of sampled stochastic differential equations	Singer, Hermann
443	2009	Konsensuale Effizienzbewertung und -verbesserung – Untersuchungen mittels der Data Envelopment Analysis (DEA)	Rödder, Wilhelm Reucher, Elmar
444	2009	Legal Uncertainty – Is Hamonization of Law the Right Answer? A Short Overview	Wagner, Helmut
445	2009	Fast Continuous-Discrete DAF-Filters	Mazzoni, Thomas
446	2010	Quantitative Evaluierung von Multi-Level Marketingsystemen	Lorenz, Marina Mazzoni, Thomas

447	2010	Quasi-Continuous Maximum Entropy Distribution Approximation with Kernel Density	Mazzoni, Thomas Reucher, Elmar
448	2010	Solving a Bi-Objective Winner Determination Problem in a Transportation Procurement Auction	Buer, Tobias Pankratz, Giselher
449	2010	Are Short Term Stock Asset Returns Predictable? An Extended Empirical Analysis	Mazzoni, Thomas
450	2010	Europäische Gesundheitssysteme im Vergleich – Effizienzmessungen von Akutkrankenhäusern mit DEA –	Reucher, Elmar Sartorius, Frank
451	2010	Patterns in Object-Oriented Analysis	Blaimer, Nicolas Bortfeldt, Andreas Pankratz, Giselher
452	2010	The Kuznets-Kaldor-Puzzle and Neutral Cross-Capital-Intensity Structural Change	Stijepic, Denis Wagner, Helmut
453	2010	Monetary Policy and Boom-Bust Cycles: The Role of Communication	Knütter, Rolf Wagner, Helmut
454	2010	Konsensuale Effizienzbewertung und –verbesserung mittels DEA – Output- vs. Inputorientierung –	Reucher, Elmar Rödder, Wilhelm
455	2010	Consistent Modeling of Risk Averse Behavior with Spectral Risk Measures	Wächter, Hans Peter Mazzoni, Thomas