# A Constructive Approach to Conway's Theory of Games *

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Abstract. John H. Conway presents in his book "On Numbers and Games" [ONAG] a general method to create a class of numbers containing all real numbers as well as every ordinal number. Using the logical law of excluded middle (LEM) he equips this class with the structure of a totally ordered field. This paper is a first step to investigate the contribution of Conway's theory to the foundations of Constructive Nonstandard Analysis. In [ONAG] Conway suggests defining real numbers as (Conway) cuts in the set of rational numbers. Following his ideas, a constructive notion of real numbers will be developed.
A constructive approach to ordinal numbers which is compatible with constructive Conway theory is presented.

*Key words.* Constructive mathematics, Conway games, Conway numbers, ordinal numbers.

## Introduction

In his book "On Numbers and Games" ([ONAG]=[11]) John Horton Conway develops a very general theory of numbers and games, frequently using the logical law of excluded middle (LEM). This paper aims to start a constructive investigation of this theory. Following the ideas of Conway, constructive notions for Conway games and Conway numbers will be developed and a constructive version of Conway's theory will be given. We shall mark any application of (LEM), constructively rejected omniscience or choice principles are avoided. Whether the author's aim to avoid even countable choice has been achieved may be judged by mathematicians with more experience in working without choice. (Fred Richman suggested to drop countable choice in his talk at the Symposion "Reuniting the Antipodes", cf. [1]; cf. also [23] and [25].)

*Conway games* are defined in Section 1 and operations of addition and subtraction for such games are presented in Section 2 resp. 3. The relations of order and equality are shown to have the expected properties in Sections 4–5. *Conway numbers* are dealt with in Section 6, and real Conway numbers are the topic of Section 7.

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Constructive background material about ordinal numbers, their order and addition is sketched in Appendix A resp. B.

# Zeroth Part
# On Games ...

## 1 Conway games

1.1 Motivation *Conway games* are played by two players (usually called *Left* and *Right*) moving alternately according to specific rules without chance moves and without hidden information. Such a game is characterized by the positions each of the two players can reach from any position with the next move. Thus, a Conway game $x$ will be described by two sets $\mathrm{L}_x$ and $\mathrm{R}_x$, the sets of *Left* resp. *Right options* (i. e. positions reachable by Left resp. Right from the starting position of $x$ within one move). As every position $P$ in a game $x$ can be identified with the *shortened game* $x_P$ (which is played according to the rules of $x$ starting from position $P$) the sets $\mathrm{L}_x$ and $\mathrm{R}_x$ will be identified with sets of Conway games. Vice versa, whenever $L$ and $R$ are sets of Conway games, we can construct a new Conway game $\{L|R\}$, in which Left may move to any element of $L$ whereas Right may move to any element of $R$. Having this in mind the following definition can be given.

1.2 Definition *(Conway games)*
For every set $X$ let $\Gamma(X) := \mathcal{P}(X) \times \mathcal{P}(X)$ be the set of pairs of subsets of X. Define $G_0 := \Gamma(\emptyset),\ G_1 := \Gamma(G_0),\ G_2 := \Gamma(G_1),\ \ldots$

$$G_\omega := \Gamma\Big(\bigcup_{k=0}^{\infty} G_k\Big),\ G_{\omega+1} := \Gamma(G_\omega),\ \text{etc.}$$

i. e. define $G_{\alpha+1} := \Gamma(G_\alpha)$ for every ordinal $\alpha$, and for any lim-ordinal $\lambda$ define $G_\lambda := \Gamma\left(\bigcup\{\, G_\alpha : \alpha \text{ contained in } \lambda \,\}\right)$. (Containment is introduced in A.3 of Appendix A, where a constructive notion of ordinal numbers is presented.)

Then $\mathrm{Ug}_j := \bigcup\{\, G_\alpha : \alpha \in \mathrm{On}_j \,\}$ may be called *$j$-th (Conway) game class* ($j \in \mathbb{N}_0$), and elements of the set $\mathrm{Ug}_\star := \bigcup\{\, G_\alpha : \alpha \in \mathrm{On}_\star \,\} = \bigcup_{j=0}^{\infty} \mathrm{Ug}_j$ are *(Conway) games*. (Conway took the name **Ug** to denote his proper class of all “unimpartial” games, i. e. games possibly favouring one of the players, cf. [ONAG]=[11] p. 78. The sets $\mathrm{On}_j$ and $\mathrm{On}_\star$ are defined in A.2 resp. A.7.)

1.3 Notation *(Left/Right Options)*
With the *projections* $\mathrm{pr}_{\mathrm{L}} : \mathrm{Ug}_\star \longrightarrow \mathrm{Ug}_\star,\ (L,R) \longmapsto L$ and $\mathrm{pr}_{\mathrm{R}} : \mathrm{Ug}_\star \longrightarrow \mathrm{Ug}_\star$, $(L,R) \longmapsto R$ we obtain two sets of games for every game $x \in \mathrm{Ug}_\star$: $\mathrm{L}_x := \mathrm{pr}_{\mathrm{L}}(x)$, the set of *Left options* in $x$, and $\mathrm{R}_x := \mathrm{pr}_{\mathrm{R}}(x)$, the set of *Right options* in $x$.
Two games are called *identical* if their sets of Left options and their sets of Right options coincide: $x \equiv y :\Longleftrightarrow \mathrm{L}_x = \mathrm{L}_y$ and $\mathrm{R}_x = \mathrm{R}_y$ ($x$ and $y$ have the *same form*). If $x \equiv (\mathrm{L}_x, \mathrm{R}_x)$ is a game, $x^{\mathrm{L}}$ will be a typical element of $\mathrm{L}_x$ (*typical Left option*) and $x^{\mathrm{R}}$ will be a typical element of $\mathrm{R}_x$ (*typical Right option*).
$\{x_1, \ldots, x_n | y_1, \ldots, y_m\}$ will abbreviate $(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_m\})$; instead of $y \equiv (\{z\}, \emptyset)$ we will write $y \equiv \{z|\}$ etc. Sometimes the expression $\{x^{\mathrm{L}} | x^{\mathrm{R}}\}$ will be used as notation for the game $x$.

1.4 Examples (The four simplest games)
(1) $\bot \equiv \{|\} \equiv (\emptyset, \emptyset)$, the *empty game*, in which both players are unable to move, is the only element of $\mathrm{Ug}_0$.
The following games are elements of $\mathrm{Ug}_1$:
(2) $1_\mathrm{L} \equiv \{\bot|\} \equiv (\{\bot\}, \emptyset)$, the *Left unit game*, in which Left has a move to $\bot$ while Right is unable to move;
(3) $1_\mathrm{R} \equiv \{|\bot\} \equiv (\emptyset, \{\bot\})$, the *Right unit game*, in which Right has a move to $\bot$ while Left is unable to move;
(4) $* \equiv \{\bot|\bot\} \equiv (\{\bot\}, \{\bot\})$, the *Nim unit game*, in which both players have a move to $\bot$. (The game Nim is described in [4].)

1.5 Convention *(Normal play convention)*
*A player unable to move* loses*, the other player is the* winner*.*
(Because of the Descending Chain Condition in A.3, no game can go on forever.)

1.6 Definition *(Outcome classes)*
For any $x \in \mathrm{Ug}_\star$ define (a game theoretic interpretation is given in 1.7)

| | | | |
|---|---|---|---|
| $x \geq 0$ | $:\Longleftrightarrow$ | $\forall\, x^\mathrm{R} \in \mathrm{R}_x : x^\mathrm{R} \rhd 0$ | (Left can win if Right starts), |
| $x \rhd 0$ | $:\Longleftrightarrow$ | $\exists\, x^\mathrm{L} \in \mathrm{L}_x : x^\mathrm{L} \geq 0$ | (Left can win if Left starts), |
| $x \leq 0$ | $:\Longleftrightarrow$ | $\forall\, x^\mathrm{L} \in \mathrm{L}_x : x^\mathrm{L} \lhd 0$ | (Right can win if Left starts), |
| $x \lhd 0$ | $:\Longleftrightarrow$ | $\exists\, x^\mathrm{R} \in \mathrm{R}_x : x^\mathrm{R} \leq 0$ | (Right can win if Right starts), |
| $x > 0$ | $:\Longleftrightarrow$ | $x \geq 0$ and $x \rhd 0$ | ($x$ is *positive*, Left can win), |
| $x < 0$ | $:\Longleftrightarrow$ | $x \leq 0$ and $x \lhd 0$ | ($x$ is *negative*, Right can win), |
| $x \parallel 0$ | $:\Longleftrightarrow$ | $x \rhd 0$ and $x \lhd 0$ | ($x$ is *fuzzy*, the first player can win), |
| $x = 0$ | $:\Longleftrightarrow$ | $x \geq 0$ and $x \leq 0$ | ($x$ is *zero*, the second player can win). |

(Here 'can win' stands for 'has a winning strategy'. Relying on this intuitive concept, some readers may prefer to define the outcome classes by the express-sions in parentheses. They can arrive at the formal definition given here by considering remark 1.7. Other readers may use the formal definitions given here to make precise the concept of *winning strategy* using remark 1.7.)

1.7 Remark Left can win $x$ in case Right moves first (i. e. $x \geq 0$) if all possible Right moves lead to games which Left can win, provided Left is allowed to make the first move there. Left can win $x$ in case Left moves first (i. e. $x \rhd 0$) if there is a Left (winning) move leading to a game which Left can win, provided Right has to move first there. The outcome classes in favour of Right ($x \leq 0$ and $x \lhd 0$) can be interpreted similarly.

1.8 Examples Here are the outcome classes for the games from 1.4:
(1) $\bot = 0$, as both players are unable to move in $\bot$;
(2) $1_\mathrm{L} > 0$, as Left wins (by moving to $\bot$ or since Right has no move);
(3) $1_\mathrm{R} < 0$, as Right wins (by moving to $\bot$ or since Left has no move);
(4) $* \parallel 0$, as the first player wins by moving to $\bot$.

1.9 Proposition
(1) *For all games $x \in \mathrm{Ug}_\star$ we have*
 (i) $\neg(x \geq 0$ and $x \lhd 0)$,
 (ii) $\neg(x \leq 0$ and $x \rhd 0)$.
(2) *The logical* law of excluded middle*,*
 (LEM) $\psi$ or $\neg\psi$ for every proposition $\psi$,
 *is equivalent to each of the following statements:*
 (a) $x \geq 0$ or $x \lhd 0$ for all $x \in \mathrm{Ug}_\star$,
 (b) $x \leq 0$ or $x \rhd 0$ for all $x \in \mathrm{Ug}_\star$.

PROOF:
(1): The proofs of (i) and (ii) are carried out by *mutual game induction*, i. e. by *game induction* (transfinite induction on $\alpha \in \mathrm{On}_\star$, $x \in G_\alpha$) for the conjunction of (i) and (ii); the induction basis will not be mentioned as there are no options of $\{|\}$, the only element of $G_0 = \Gamma(\emptyset)$.
*Ind. Step:* (i) Suppose we have $x \geq 0$ and $x \lhd 0$, hence $x^{\mathrm{R}} \rhd 0$ for all $x^{\mathrm{R}} \in \mathrm{R}_x$ and $x^{\mathrm{R}} \leq 0$ for some $x^{\mathrm{R}} \in \mathrm{R}_x$; but $x^{\mathrm{R}} \rhd 0$ and $x^{\mathrm{R}} \leq 0$ would contradict Ind. Hyp. (ii).
(ii) $x \leq 0$ and $x \rhd 0$ would yield analogously a contradiction to Ind. Hyp. (i).
(2): "(LEM) $\Longrightarrow$ (a),(b)" is also proved by mutual game induction:
Since $\neg(x \lhd 0) \Longrightarrow \neg(\exists\, x^{\mathrm{R}} \in \mathrm{R}_x : x^{\mathrm{R}} \leq 0) \Longrightarrow \forall\, x^{\mathrm{R}} \in \mathrm{R}_x : \neg(x^{\mathrm{R}} \leq 0) \Longrightarrow \forall\, x^{\mathrm{R}} \in \mathrm{R}_x : x^{\mathrm{R}} \rhd 0$ [by Ind. Hyp. (b)] $\Longrightarrow x \geq 0$, we can deduce (a) from (LEM) via ( $x \lhd 0$ or $\neg(x \lhd 0)$ ); (b) is deduced similarly from (LEM) and Ind. Hyp. (a).
"(a) $\Longrightarrow$ (LEM)":
Let $x_\psi \equiv (\emptyset, \{\, x \in \{\bot\} : \psi \,\}) \in \mathrm{Ug}_\star$. Then with $\mathrm{R}_\psi := \{\, x \in \{\bot\} : \psi \,\}$ we have

$x_\psi \geq 0 \Longleftrightarrow \forall\, y \in \mathrm{R}_\psi : y \rhd 0 \Longleftrightarrow \bot \notin \{\, x \in \{\bot\} : \psi \,\} \Longleftrightarrow \neg\psi$ and

$x_\psi \lhd 0 \Longleftrightarrow \exists\, y \in \mathrm{R}_\psi : y \leq 0 \Longleftrightarrow \bot \in \{\, x \in \{\bot\} : \psi \,\} \Longleftrightarrow \psi$.

"(b) $\Longrightarrow$ (LEM)" is proved analogously. □

# 2 Addition

2.1 MOTIVATION Two games $x, y \in \mathrm{Ug}_\star$ can be played simultaneously by the *simultaneous play rule*: The player to move may choose from the allowed moves in exactly one of the components $x$ or $y$ leaving the other component unchanged. This leads to the following inductive definition of the sum $x + y$ of two games.

2.2 DEFINITION *(Addition)*
For games $x, y \in \mathrm{Ug}_\star$ their *(disjunctive) sum* is given by $x + y \equiv (\mathrm{L}_{x+y}, \mathrm{R}_{x+y})$ with $\mathrm{L}_{x+y} := (\mathrm{L}_x + y) \cup (x + \mathrm{L}_y) = \{\, x^{\mathrm{L}} + y : x^{\mathrm{L}} \in \mathrm{L}_x \,\} \cup \{\, x + y^{\mathrm{L}} : y^{\mathrm{L}} \in \mathrm{L}_y \,\}$ and $\mathrm{R}_{x+y} := (\mathrm{R}_x + y) \cup (x + \mathrm{R}_y) = \{\, x^{\mathrm{R}} + y : x^{\mathrm{R}} \in \mathrm{R}_x \,\} \cup \{\, x + y^{\mathrm{R}} : y^{\mathrm{R}} \in \mathrm{R}_y \,\}$.
Or, using a more condensed notation, $x + y \equiv \left\{x^{\mathrm{L}} + y, x + y^{\mathrm{L}} \middle| x^{\mathrm{R}} + y, x + y^{\mathrm{R}}\right\}$.

2.3 REMARK For all $\alpha, \beta \in \mathrm{On}_j$, $j \in \mathbb{N}_0$, there is $\gamma \in \mathrm{On}_j$ with $x + y \in G_\gamma$ for all $x \in G_\alpha$ and $y \in G_\beta$. The function $\mathrm{add}_{\mathrm{Ug}_\star} : \mathrm{Ug}_\star \times \mathrm{Ug}_\star \longrightarrow \mathrm{Ug}_\star$, $(x, y) \longmapsto x + y$ satisfies $\mathrm{add}_{\mathrm{Ug}_\star}(x, y) \in \mathrm{Ug}_j$ whenever $x, y \in \mathrm{Ug}_j$, $j \in \mathbb{N}_0$.

2.4 EXAMPLES
(1) $\bot + \bot \equiv \{|\} + \{|\} \equiv \{|\} \equiv \bot$;
(2) $1_{\mathrm{L}} + \bot \equiv \{\bot|\} + \{|\} \equiv \{\bot + \bot|\} \equiv 1_{\mathrm{L}}$ by (1);
(3) $\bot + 1_{\mathrm{R}} \equiv 1_{\mathrm{R}}$ can be seen similarly;
(4) $1_{\mathrm{L}} + 1_{\mathrm{R}} \equiv \{\bot + 1_{\mathrm{R}} | 1_{\mathrm{L}} + \bot\} \equiv \{1_{\mathrm{R}} | 1_{\mathrm{L}}\}$ by (2) and (3).

2.5 PROPOSITION ($\mathrm{Ug}_\star$ monoid)
$\mathrm{Ug}_\star$ *is a* commutative monoid *with neutral element* $\bot \equiv \{|\}$:
*For all games* $x, y, z \in \mathrm{Ug}_\star$ *we have*
(1) $x + \bot \equiv x$,
(2) $x + y \equiv y + x$,
(3) $(x + y) + z \equiv x + (y + z)$.

PROOF: The proofs are carried out by ordinary game inductions:

(1) : $x + \bot \equiv \{x^{\mathrm{L}} + \bot \mid x^{\mathrm{R}} + \bot\} \equiv \{x^{\mathrm{L}} \mid x^{\mathrm{R}}\}$ [Ind. Hyp.] $\equiv x$.

(2) : $y + x \equiv \{y^{\mathrm{L}} + x, y + x^{\mathrm{L}} \mid y^{\mathrm{R}} + x, y + x^{\mathrm{R}}\}$
$\equiv \{x + y^{\mathrm{L}}, x^{\mathrm{L}} + y \mid x + y^{\mathrm{R}}, x^{\mathrm{R}} + y\}$ [Ind. Hyp.] $\equiv x + y$.

(3) : $(x + y) + z \equiv \{(x^{\mathrm{L}} + y) + z, (x + y^{\mathrm{L}}) + z, (x + y) + z^{\mathrm{L}} \mid \ldots\}$
$\equiv \{x^{\mathrm{L}} + (y + z), x + (y^{\mathrm{L}} + z), x + (y + z^{\mathrm{L}}) \mid \ldots\}$ [Ind. Hyp.]
$\equiv x + (y + z)$. □

2.6 LEMMA (Outcome classes and addition)

*For all games $x, y \in \mathrm{Ug}_\star$ the following statements hold.*

(1) $x \geq 0$ and $y \geq 0 \Longrightarrow x + y \geq 0$,
(2) $x \geq 0$ and $y \rhd 0 \Longrightarrow x + y \rhd 0$,
(3) $x + y \geq 0$ and $y \leq 0 \Longrightarrow x \geq 0$,
(4) $x + y \geq 0$ and $y \lhd 0 \Longrightarrow x \rhd 0$,
(5) $x + y \rhd 0$ and $y \leq 0 \Longrightarrow x \rhd 0$.

(A game theoretic interpretation of some of these implications is given in 2.7.)

PROOF:

(1) and (2) are mutually proved by a straightforward game induction.

(3), (4) and (5) are also proved by mutual game induction (cf. 1.9):

(3) : $x + y \geq 0 \Longrightarrow \forall x^{\mathrm{R}} \in \mathrm{R}_x : x^{\mathrm{R}} + y \rhd 0$
$\Longrightarrow \forall x^{\mathrm{R}} \in \mathrm{R}_x : x^{\mathrm{R}} \rhd 0$ [by Ind. Hyp. (5)].

(4) : $x + y \geq 0 \Longrightarrow \forall y^{\mathrm{R}} \in \mathrm{R}_y : x + y^{\mathrm{R}} \rhd 0$
$\Longrightarrow x \rhd 0$ [by Ind. Hyp. (5)],
because $y \lhd 0 \Longrightarrow \exists y^{\mathrm{R}} \in \mathrm{R}_y : y^{\mathrm{R}} \leq 0$.

(5) : $x + y \rhd 0 \Longrightarrow \exists x^{\mathrm{L}} \in \mathrm{L}_x : x^{\mathrm{L}} + y \geq 0$ or $\exists y^{\mathrm{L}} \in \mathrm{L}_y : x + y^{\mathrm{L}} \geq 0$;
first case: $\exists x^{\mathrm{L}} \in \mathrm{L}_x : x^{\mathrm{L}} + y \geq 0 \Longrightarrow \exists x^{\mathrm{L}} \in \mathrm{L}_x : x^{\mathrm{L}} \geq 0$ [by Ind. Hyp. (3)];
second case: $\exists y^{\mathrm{L}} \in \mathrm{L}_y : x + y^{\mathrm{L}} \geq 0 \Longrightarrow x \rhd 0$ [by Ind. Hyp. (4)],
because $y \leq 0 \Longrightarrow \forall y^{\mathrm{L}} \in \mathrm{L}_y : y^{\mathrm{L}} \lhd 0$. □

2.7 INTERPRETATION

The *first implication* of 2.6 asserts that Left can win the sum if Right starts, provided Left can win each component. Indeed, Left can find a good reply to any move of Right because there is a good reply in any component, thus Left will win by choosing always the same component as Right and playing a winning move there.

The *second implication* of 2.6 means that Left having the first move can win the sum $x + y$, provided Left can win one component $x$ with Right moving first and the other component $y$ having the first move. Indeed, Left may start with a move from $x + y$ to $x + y^{\mathrm{L}} \geq 0$ choosing a winning move $y^{\mathrm{L}} \geq 0$ in $y$.

The *third implication* of 2.6 says that Left can win a game $x$ if Right starts, provided Left can win a sum $x + y$ if Right starts, where $y$ is a game which can be won by Right having the second move. Indeed, Right may choose to move in $x$ and, as $y \leq 0$, every reply of Left in $y$ can be countered by Right with a good move. Thus, Left can win the sum $x + y$ only by finding a winning move in the game $x$.

The remaining implications of 2.6 can be interpreted in an analogous manner.

# 3 Subtraction

3.1 Motivation The *antigame* $-x$ is played like the original game $x$ in which the roles of Left and Right have been interchanged: The allowed moves for Left in $-x$ correspond to the Right moves in $x$ and the allowed moves for Right in $-x$ correspond to the Left moves in $x$, where the roles have to be interchanged in the options too. The following definition formalizes this idea.

3.2 Definition *(Subtraction)*
For every game $x \in \mathrm{Ug}_\star$ its *antigame* is given by $-x \equiv (\mathrm{L}_{-x}, \mathrm{R}_{-x})$ with
$\mathrm{L}_{-x} := -\mathrm{R}_x = \left\{ -x^{\mathrm{R}} : x^{\mathrm{R}} \in \mathrm{R}_x \right\}$ and $\mathrm{R}_{-x} := -\mathrm{L}_x = \left\{ -x^{\mathrm{L}} : x^{\mathrm{L}} \in \mathrm{L}_x \right\}$.
Or, using a more condensed notation, $-x \equiv \left\{-x^{\mathrm{R}} \middle| -x^{\mathrm{L}}\right\}$.
The *difference* of $x, y \in \mathrm{Ug}_\star$ is defined to be

$$x - y \equiv x + (-y) \equiv \left\{x^{\mathrm{L}} - y, x - y^{\mathrm{R}} \middle| x^{\mathrm{R}} - y, x - y^{\mathrm{L}}\right\}.$$

3.3 Remark
We have $-x \in \mathrm{Ug}_j$ whenever $x \in \mathrm{Ug}_j$, $j \in \mathbb{N}_0$, so using $\mathrm{add}_{\mathrm{Ug}_\star}$ from 2.3 we obtain a function $\mathrm{sub}_{\mathrm{Ug}_\star} : \mathrm{Ug}_\star \times \mathrm{Ug}_\star \longrightarrow \mathrm{Ug}_\star$, $(x, y) \longmapsto x - y$ satisfying $\mathrm{sub}_{\mathrm{Ug}_\star}(x, y) \in \mathrm{Ug}_j$ whenever $x, y \in \mathrm{Ug}_j$, $j \in \mathbb{N}_0$.

3.4 Examples
(1) $-\bot \equiv -\{|\} \equiv \{|\} \equiv \bot$;
(2) $-1_{\mathrm{L}} \equiv -\{\bot|\} \equiv \{|-\bot\} \equiv 1_{\mathrm{R}}$ by (1);
(3) $-* \equiv -\{\bot|\bot\} \equiv \{-\bot|-\bot\} \equiv *$ by (1);
(4) $1_{\mathrm{L}} - 1_{\mathrm{L}} \equiv 1_{\mathrm{L}} + 1_{\mathrm{R}} \equiv \{1_{\mathrm{R}}|1_{\mathrm{L}}\}$ by (2) and 2.4 (4).

3.5 Note *For all games $x \in \mathrm{Ug}_\star$ we have*
(1) $x \le 0 \iff -x \ge 0$,
(2) $x \lhd 0 \iff -x \rhd 0$,
(3) $x < 0 \iff -x > 0$.
((1) and (2) are proved by mutual game induction, then (3) follows.)

3.6 Proposition
*For all games $x, y \in \mathrm{Ug}_\star$ the following statements hold.*
(1) $-(-x) \equiv x$,
(2) $-(x + y) \equiv (-x) + (-y)$,
(3) $-(x - y) \equiv y - x$,
(4) $x - x = 0$.
(Example 3.4 (4) shows that (4) cannot be replaced by $x - x \equiv \bot$.)

Proof:
The proofs of (1) and (2) are carried out by ordinary game inductions:
(1) $-(-x) \equiv -\left\{-x^{\mathrm{R}} \middle| -x^{\mathrm{L}}\right\} \equiv \left\{-(-x^{\mathrm{L}}) \middle| -(-x^{\mathrm{R}})\right\}$ [Ind. Hyp.] $\equiv x$ .
(2) $-(x + y) \equiv \left\{-(x^{\mathrm{R}} + y), -(x + y^{\mathrm{R}}) \middle| -(x^{\mathrm{L}} + y), -(x + y^{\mathrm{L}})\right\}$
$\equiv \left\{(-x^{\mathrm{R}}) + (-y), (-x) + (-y^{\mathrm{R}}) \middle| (-x^{\mathrm{L}}) + (-y), (-x) + (-y^{\mathrm{L}})\right\}$ [Ind. Hyp.]
$\equiv (-x) + (-y)$.
(3) is a consequence of (1), (2) and 2.5 (2).
(4): We prove $x - x \le 0$. (This together with (3) and 3.5 (1) yields $x - x \ge 0$.) Because of $x^{\mathrm{L}} - x^{\mathrm{L}} \le 0$ [Ind. Hyp.] we have $x^{\mathrm{L}} - x \lhd 0$ for all $x^{\mathrm{L}} \in \mathrm{L}_x$, and because of $x^{\mathrm{R}} - x^{\mathrm{R}} \le 0$ [Ind. Hyp.] we have $x - x^{\mathrm{R}} \lhd 0$ for all $x^{\mathrm{R}} \in \mathrm{R}_x$. Hence $(x - x)^{\mathrm{L}} \lhd 0$ holds for all $(x - x)^{\mathrm{L}} \in \mathrm{L}_{x-x}$, i. e. $x - x \le 0$. □

# 4 Order

4.1 Definition *(Order)*
For games $x, y \in \mathrm{Ug}_\star$ define

$x \geq y \;:\Longleftrightarrow\; x - y \geq 0$ ($x$ is *at least as favourable* for Left as $y$),
$x \leq y \;:\Longleftrightarrow\; x - y \leq 0$ ($x$ is *at most as favourable* for Left as $y$),
$x \rhd y \;:\Longleftrightarrow\; x - y \rhd 0$ ($x$ is *partly more favourable* for Left than $y$),
$x \lhd y \;:\Longleftrightarrow\; x - y \lhd 0$ ($x$ is *partly less favourable* for Left than $y$),
$x > y \;:\Longleftrightarrow\; x - y > 0$ ($x$ is *more favourable* for Left than $y$),
$x < y \;:\Longleftrightarrow\; x - y < 0$ ($x$ is *less favourable* for Left than $y$),
$x = y \;:\Longleftrightarrow\; x - y = 0$ ($x$ and $y$ are *equally favourable* for Left),
$x \parallel y \;:\Longleftrightarrow\; x - y \parallel 0$ ($x$ and $y$ are *incompatible*).

4.2 Hint *For all games $x, y \in \mathrm{Ug}_\star$ we have*
(1) $x \leq y \iff y \geq x \iff -x \geq -y$,
(2) $x \lhd y \iff y \rhd x \iff -x \rhd -y$,
(3) $x < y \iff y > x \iff -x > -y$.
(The proofs are straightforward with 3.5 and 3.6.)

4.3 Lemma (Characterization of order)
*For all games $x, y \in \mathrm{Ug}_\star$ the following statements hold.*
(1) $x \leq y \iff \forall\, x^{\mathrm{L}} \in \mathrm{L}_x : x^{\mathrm{L}} \lhd y$ and $\forall\, y^{\mathrm{R}} \in \mathrm{R}_y : x \lhd y^{\mathrm{R}}$,
(2) $x \lhd y \iff \exists\, x^{\mathrm{R}} \in \mathrm{R}_x : x^{\mathrm{R}} \leq y$ or $\exists\, y^{\mathrm{L}} \in \mathrm{L}_y : x \leq y^{\mathrm{L}}$,
(3) $x > y \iff x \geq y$ and $x \rhd y$,
(4) $x = y \iff x \geq y$ and $y \geq x$,
(5) $x \parallel y \iff x \rhd y$ and $y \rhd x$.

Proof:
(1): $x \leq y \iff \forall\, (x-y)^{\mathrm{L}} \in \mathrm{L}_{x-y} : (x-y)^{\mathrm{L}} \lhd 0$
$\iff \forall\, x^{\mathrm{L}} \in \mathrm{L}_x : x^{\mathrm{L}} - y \lhd 0$ and $\forall\, y^{\mathrm{R}} \in \mathrm{R}_y : x - y^{\mathrm{R}} \lhd 0$.
(2) is proved similarly, while (3), (4) and (5) are plain. □

4.4 Note (Properties of order)
*For all games $x, y, z \in \mathrm{Ug}_\star$ the following statements hold.*
(1) $x \geq x$,
(2) $x \geq y$ and $y \geq z \implies x \geq z$,
(3) $x \geq y$ and $y \rhd z \implies x \rhd z$,
(4) $x \geq y \iff x + z \geq y + z$,
(5) $x \rhd y \iff x + z \rhd y + z$,
(6) $x^{\mathrm{L}} \lhd x$ and $x \lhd x^{\mathrm{R}}$ whenever $x^{\mathrm{L}} \in \mathrm{L}_x$, $x^{\mathrm{R}} \in \mathrm{R}_x$.
(For (1) and (6) use 3.6 (4), for (2), (3), (4) and (5) use 2.6.)

4.5 Remark (Properties of strict order)
*For all games $x, y, z \in \mathrm{Ug}_\star$ we have*
(1) $\neg(x > x)$,
(2) $x > y \implies \neg(y > x)$,
(3) $x > y$ and $y > z \implies x > z$,
(4) $x > y \iff x + z > y + z$.
((1) and (2) are proved with 1.9 (1), while (3) and (4) are consequences of 4.4.)

4.6 Result $\geq$ is a *preorder relation* (reflexive and transitive) on $\mathrm{Ug}_\star$ with associated equivalence relation $=$, and $>$ is a strict partial order relation on $\mathrm{Ug}_\star$.

# 5 Equality

5.1 Note The equivalence relation $=$ (cf. 4.1 and 4.6) is *invariant with respect to translations and reflections*, that is to say that for all $x, y, z \in \mathrm{Ug}_\star$ we have

(1) $x = y \Longrightarrow x + z = y + z$,
(2) $x = y \Longrightarrow -x = -y$.

(The proofs are straightforward with 4.3 (4), 4.4 and 3.5.)

5.2 Observation Addition and subtraction preserve equality, and $\geq$, $\rhd$, $>$ as well as $\|$ allow substitution of equals, viz. for all $x_1, x_2, y_1, y_2, x, y, z \in \mathrm{Ug}_\star$ the following statements hold.

(1) $x_1 = x_2$ and $y_1 = y_2 \implies x_1 + y_1 = x_2 + y_2$,
(2) $x_1 = x_2$ and $y_1 = y_2 \implies x_1 - y_1 = x_2 - y_2$,
(3) $x = y$ and $y \varrho z \implies x \varrho z$ whenever $\varrho \in \{\geq, \rhd, >, \|\}$,
(4) $x \varrho y$ and $y = z \implies x \varrho z$ whenever $\varrho \in \{\geq, \rhd, >, \|\}$.

((1) and (2) are consequences of 5.1, the proofs of (3) and (4) for $\geq$ and $\rhd$ are plain with 4.4, then (3) and (4) for $>$ and $\|$ follow.)

5.3 Result $\mathbf{Ug}_\star$ (i. e. $\mathrm{Ug}_\star$ modulo $=$) is a partially ordered group.
(Here bold print symbolizes the employment of $=$ as equivalence relation.)

5.4 Lemma
*Let $x, y \in \mathrm{Ug}_\star$ be games and let $L', R' \subseteq G_\alpha$ be sets of games, $\alpha \in \mathrm{On}_\star$. Then*

(1) $(\mathrm{L}_x \cup L', \mathrm{R}_x) = x$, *if $L' \lhd x$ (i. e. if $x' \lhd x$ for all $x' \in L'$);*
(2) $(\mathrm{L}_y, \mathrm{R}_y \cup R') = y$, *if $y \lhd R'$ (i. e. if $y \lhd y'$ for all $y' \in R'$);*
(3) $(\mathrm{L}_x \cup L', \mathrm{R}_x \cup R') = x$, *if $L' \lhd x \lhd R'$*
*(i. e. if $x' \lhd x \lhd y'$ for all $x' \in L'$, $y' \in R'$).*

Proof:
In (1) let $\tilde{x} \equiv (\mathrm{L}_x \cup L', \mathrm{R}_x)$ and prove $\tilde{x} \leq x$: We have $L' \lhd x$ by assumption and $\mathrm{L}_x \lhd x$ by 4.4 (6), hence $\mathrm{L}_{\tilde{x}} = (\mathrm{L}_x \cup L') \lhd x$. Moreover, by 4.4 (6), we have $\tilde{x} \lhd \mathrm{R}_{\tilde{x}} = \mathrm{R}_x$. ($x \leq \tilde{x}$ is proved similarly.)
For (2) apply (1) with $x = -y$ and $L' = -R'$, for (3) apply (1) and (2). □

5.5 Notation
For every subset $A \subseteq X$ of any set $X$ with preorder $\leq$ we call

$\uparrow\! A := \{\, x \in X : \exists\, a \in A : a \leq x \,\}$ the *upwards closure* of $A$,
$\downarrow\! A := \{\, x \in X : \exists\, a \in A : x \leq a \,\}$ the *downwards closure* of $A$.

5.6 Hint $\uparrow$ and $\downarrow$ are *closure operators*:

(1) $A \subseteq \uparrow\! A$ and $A \subseteq \downarrow\! A$,
(2) $\uparrow\uparrow\! A = \uparrow\! A$ and $\downarrow\downarrow\! A = \downarrow\! A$,
(3) $A \subseteq B \Longrightarrow \uparrow\! A \subseteq \uparrow\! B$ and $\downarrow\! A \subseteq \downarrow\! B$,
(4) $(A \subseteq \uparrow\! B \iff \uparrow\! A \subseteq \uparrow\! B)$ and $(A \subseteq \downarrow\! B \iff \downarrow\! A \subseteq \downarrow\! B)$.

(The proofs are plain; *closure spaces* are presented in [17].)

5.7 Proposition *For all games $x, y \in \mathrm{Ug}_\star$ the following statements hold.*

(1) $\mathrm{L}_x \subseteq \downarrow\!\mathrm{L}_y$ and $\mathrm{R}_y \subseteq \uparrow\!\mathrm{R}_x \Longrightarrow x \leq y$,
(2) $\downarrow\!\mathrm{L}_x = \downarrow\!\mathrm{L}_y$ and $\uparrow\!\mathrm{R}_y = \uparrow\!\mathrm{R}_x \Longrightarrow x = y$.

(It is not possible to replace the implications by equivalences. For instance, $x \equiv \{*|\} = \{|\} \equiv y$ by 5.4 as $* \equiv \{\perp|\perp\} \lhd \perp \equiv \{|\}$, but $\mathrm{L}_x = \{*\} \neq \emptyset = \downarrow\!\mathrm{L}_y$.)

PROOF:
(1): Because of $\mathrm{L}_x \subseteq {\downarrow}\mathrm{L}_y$ we have $\mathrm{L}_x \lhd y$ (i. e. $x^{\mathrm{L}} \lhd y$ for all $x^{\mathrm{L}} \in \mathrm{L}_x$) and because of $\mathrm{R}_y \subseteq {\uparrow}\mathrm{R}_x$ we have $x \lhd \mathrm{R}_y$ (i. e. $x \lhd y^{\mathrm{R}}$ for all $y^{\mathrm{R}} \in \mathrm{R}_y$).
(2) is a consequence of (1). □

5.8 INTERPRETATION Proposition 5.7 (2) can be interpreted as follows:
*The omission of any dominated option leaves the value of a game unchanged.*
(A *dominated* option of $x$ is a Left option $x^{\mathrm{L}} \in {\downarrow}(\mathrm{L}_x \setminus \{x^{\mathrm{L}}\})$ or a Right option $x^{\mathrm{R}} \in {\uparrow}(\mathrm{R}_x \setminus \{x^{\mathrm{R}}\})$.)

# First Part
# . . . and Numbers

## 6 Conway numbers

6.1 MOTIVATION The difference $x^{\mathrm{L}} - x$ (resp. $x - x^{\mathrm{R}}$) is the so-called *incentive* for a move from $x$ to $x^{\mathrm{L}}$ (resp. from $x$ to $x^{\mathrm{R}}$), cf. [ONAG]=[11] p. 207.
If $x^{\mathrm{L}} < x$ (resp. $x^{\mathrm{R}} > x$) does always hold, Left (resp. Right) will try to avoid moving in $x$ because every move would be disadvantageous. (High values are advantageous for Left, while low values are advantageous for Right.)
A game $z \in \mathrm{Ug}_\star$ with this *negative incentive property*, in which in addition all (Left and Right) options also have this same property, is called *Conway number.*

6.2 DEFINITION *(Conway numbers)*
A *Conway number* is a Conway game $z \in \mathrm{Ug}_\star$ satisfying the following conditions:
(N1) $z^{\mathrm{L}} \lhd z^{\mathrm{R}}$ for all $z^{\mathrm{L}} \in \mathrm{L}_z$, $z^{\mathrm{R}} \in \mathrm{R}_z$,
(N2) all $z^{\mathrm{L}} \in \mathrm{L}_z$ and all $z^{\mathrm{R}} \in \mathrm{R}_z$ are Conway numbers.
Analogously to 1.2 define sets $N_\alpha \subset G_\alpha$ for every $\alpha \in \mathrm{On}_\star$:

$$N_0 := \Gamma(\emptyset),\ N_1 := \{\, z \in \Gamma(N_0) : \text{(N1)} \,\},\ N_2 := \{\, z \in \Gamma(N_1) : \text{(N1)} \,\},\ \ldots$$

$$N_\omega := \left\{ z \in \Gamma\left(\bigcup_{k=0}^{\infty} G_k\right) : \text{(N1)} \right\},\ N_{\omega+1} := \{\, z \in \Gamma(G_\omega) : \text{(N1)} \,\},\ \text{etc.}$$

Then $\mathrm{No}_j := \bigcup\{\, N_\alpha : \alpha \in \mathrm{On}_j \,\}$ may be called the *j-th Conway number class* ($j \in \mathbb{N}_0$), and $\mathrm{No}_\star := \bigcup\{\, N_\alpha : \alpha \in \mathrm{On}_\star \,\} = \bigcup_{j=0}^{\infty} \mathrm{No}_j$ denotes the set of all *Conway numbers*. (Conway took the name **No** to denote his proper class of numbers, cf. [ONAG]=[11] p. 4.)

6.3 EXAMPLES
$(M, \emptyset)$ and $(\emptyset, M)$ are Conway numbers for any set of Conway numbers $M \subset N_\alpha$ ($\alpha \in \mathrm{On}_\star$). Especially $\bot \equiv \{|\}$, $1_{\mathrm{L}} \equiv \{\bot|\}$ and $1_{\mathrm{R}} \equiv \{|\bot\}$ are Conway numbers.
$* \equiv \{\bot|\bot\}$ is not a Conway number as $\bot \lhd \bot$ does not hold.

6.4 REMARK (Properties of Conway numbers)
*For all Conway numbers $z, z_1, z_2 \in \mathrm{No}_\star$ we have*
(1) $z^{\mathrm{L}} < z < z^{\mathrm{R}}$ whenever $z^{\mathrm{L}} \in \mathrm{L}_z$, $z^{\mathrm{R}} \in \mathrm{R}_z$;
(2) $z_1 \lhd z_2 \iff z_1 < z_2$;
(3) $-z, z_1 + z_2 \in \mathrm{No}_\star$.
Therefore $\mathbf{No}_\star$, i. e. $\mathrm{No}_\star$ modulo $=$, is a subgroup of $\mathbf{Ug}_\star$.
( (1) is proved using (N1) and 4.4 (3) by *Conway number induction*, i. e. by transfinite induction on $\alpha \in \mathrm{On}_\star$, $z \in N_\alpha$; with 4.3 (2) and 4.4 we obtain (2) as consequence of (1) , and (3) is proved using (2) by Conway game inductions.)

6.5 Lemma (Simplicity Lemma)
*Let* $x \in \mathrm{Ug}_\star$ *be a game. Then* $x = z$ *for any number* $z \in \mathrm{No}_\star$ *with* $\mathrm{L}_x \lhd z \lhd \mathrm{R}_x$ *(i. e. with* $x^{\mathrm{L}} \lhd z \lhd x^{\mathrm{R}}$ *for all* $x^{\mathrm{L}} \in \mathrm{L}_x$, $x^{\mathrm{R}} \in \mathrm{R}_x$*),* $\mathrm{L}_z \subseteq {\downarrow}\mathrm{L}_x$ *and* $\mathrm{R}_z \subseteq {\uparrow}\mathrm{R}_x$*.*
(If the assumptions hold, $z$ is the *"simplest"* Conway number between $\mathrm{L}_x$ and $\mathrm{R}_x$, because – as $z^{\mathrm{L}} \le x^{\mathrm{L}}$ and $x^{\mathrm{R}} \le z^{\mathrm{R}}$ hold for some $x^{\mathrm{L}} \in \mathrm{L}_x$ and $x^{\mathrm{R}} \in \mathrm{R}_x$ – it is impossible that $x^{\mathrm{L}} \lhd z^{\mathrm{L}} \lhd x^{\mathrm{R}}$ or $x^{\mathrm{L}} \lhd z^{\mathrm{R}} \lhd x^{\mathrm{R}}$ can always hold.)

Proof:
$z \le x$: Because of $\mathrm{L}_z \subseteq {\downarrow}\mathrm{L}_x$ we have $\mathrm{L}_z \lhd x$ (i. e. $z^{\mathrm{L}} \lhd x$ for all $z^{\mathrm{L}} \in \mathrm{L}_z$), and by assumption we have $z \lhd \mathrm{R}_x$. ($x \le z$ is proved similarly.) □

6.6 Auxiliary Theorem (Dyadic Conway numbers)
*Set* $D_k := 2^{-k}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$*, and let* $D_\infty := \bigcup_{k=0}^{\infty} D_k$ *denote the set of* dyadic rationals (*i. e. rationals* $\frac{m}{2^k}$ *with* $m \in \mathbb{Z}$, $k \in \mathbb{N}_0$). *Then the following statements hold.*

(1) *There is a* unique map $c_\infty : D_\infty \longrightarrow \mathrm{No}_\star$ *with image* $c_\infty[D_\infty] \subseteq \mathrm{No}_1$ *and*
- (i) $c_\infty(0) \equiv \{|\} \equiv \perp$, *the neutral element of* $\mathrm{No}_\star$,
- (ii) $c_\infty(n) \equiv \{c_\infty(n-1)|\}$ *for all* $n \in \mathbb{N}$,
- (iii) $c_\infty(-n) \equiv \{|-c_\infty(n-1)\}$ for all $n \in \mathbb{N}$,
- (iv) $c_\infty\big(\frac{2\ell+1}{2^k}\big) \equiv \big\{c_\infty\big(\frac{\ell}{2^{k-1}}\big) \big| c_\infty\big(\frac{\ell+1}{2^{k-1}}\big)\big\}$ *for all* $k \in \mathbb{N}$, $\ell \in \mathbb{Z}$.

(2) $c_\infty$ *is* reflection preserving *and* strictly increasing, *i. e. we have* $c_\infty(-s) \equiv -c_\infty(s)$ *and* $c_\infty(s) < c_\infty(t)$ *if* $s < t$ *and* $s, t \in D_\infty$*.*

(3) $\overline{c}_\infty := \overline{\mathrm{pr}} \circ c_\infty$ *defines an* injective group homomorphism $\overline{c}_\infty : D_\infty \longrightarrow \mathbf{No}_\star$, *where* $\overline{\mathrm{pr}} : \mathrm{No}_\star \longrightarrow \mathbf{No}_\star$ *is the canonical projection* (with $\mathbf{No}_\star$ as in 6.4).

Proof:
(1): First construct $c_0 : \mathbb{Z} \longrightarrow \mathrm{No}_\star$ satisfying $c_0[\mathbb{Z}] \subset \mathrm{No}_1$ as well as (i), (ii) and (iii) with $c_0$ instead of $c_\infty$, then extend every map $c_{k-1} : D_{k-1} \longrightarrow \mathrm{No}_\star$ to a map $c_k : D_k \longrightarrow \mathrm{No}_\star$ using (iv) with $c_k$ instead of $c_\infty$. Finally define $c_\infty(s)$ to be $c_k(s)$ if $s \in D_k$. Uniqueness is proved by inductions on $n$ and on $k$.
(2) is proved using the identities from (1), while (3) is proved with 6.5. □

6.7 Convention
Dyadic rationals can be interpreted as Conway numbers by dint of 6.6. Numbers of the form $c_\infty(s)$ with $s \in D_\infty$ may be called *dyadic Conway numbers.*
Occurrences of $c_\infty$ will usually be suppressed, provided that no misunderstandings are to be expected, e. g.
$0 \equiv \{|\}$, $1 \equiv \{0|\}$, $2 \equiv \{1|\}$, $3 \equiv \{2|\}$, $\ldots$, $\frac{1}{2} \equiv \{0|1\}$, $\frac{1}{4} \equiv \{0|\frac{1}{2}\}$, $\frac{3}{4} \equiv \{\frac{1}{2}|1\}$,
$-1 \equiv \{|0\}$, $-2 \equiv \{|-1\}$, $-3 \equiv \{|-2\}$, $\ldots$, $-\frac{1}{2} \equiv \{-1|0\}$, $-\frac{1}{4} \equiv \{-\frac{1}{2}|0\}$ etc.
(Writing 0 for $\perp$ is consistent with Definition 1.6, as $x\varrho\perp$ is equivalent to $x\varrho 0$ for every $\varrho \in \{\ge, \rhd, \le, \lhd, >, <, \|, =\}$; in 1.4 we have $1_{\mathrm{L}} \equiv 1$ and $1_{\mathrm{R}} \equiv -1$.)

# 7 Real Conway numbers

7.1 Definition *(Real Conway numbers)*
A Conway number $z \in \mathrm{No}_\star$ is called *real* if it satisfies the following conditions:
- (R1) $-n < z < n$ for some $n \in \mathbb{N}$, and
- (R2) for all $z^{\mathrm{L}} \in \mathrm{L}_z$ there is an $m \in \mathbb{N}$ with $z - z^{\mathrm{L}} > 2^{-m}$
  and for all $z^{\mathrm{R}} \in \mathrm{R}_x$ there is an $m \in \mathbb{N}$ with $z^{\mathrm{R}} - z > 2^{-m}$.

We call a (real) Conway number $z$ *located* if $c(s) < z$ or $z < c(t)$ holds whenever $s, t \in D_\infty$ with $s < t$. In addition we set $\mathrm{No}_{\mathrm{real}} := \{ z \in \mathrm{No}_\star : z \text{ is real} \}$ and $\mathbf{R} := \overline{\mathrm{pr}}\,[\{ z \in \mathrm{No}_{\mathrm{real}} : z \text{ is located} \}]$ (with $\overline{\mathrm{pr}}$ from 6.6).

7.2 Example All dyadic Conway numbers (cf. 6.7) are located reals.

7.3 Remark (Properties of real Conway numbers)
(1) $\mathbf{R}$ is a subgroup of $\mathbf{No}_\star$.
(2) A real Conway number $z$ is located if and only if for every $k \in \mathbb{N}$ there are $s, t \in D_\infty$ with $z - 2^{-k} < s < z < t < z + 2^{-k}$.
(3) For all $z_1, z_2 \in \mathrm{No}_{\mathrm{real}}$ with $z_1 < z_2$ there is an $m \in \mathbb{N}$ with $z_2 - z_1 > 2^{-m}$.

7.4 Auxiliary Theorem (Rational Conway numbers)
(1) $c : \mathbb{Q} \longrightarrow \mathrm{No}_\star,\ q \longmapsto (c_\infty[\{ s \in D_\infty : s < q \}], c_\infty[\{ s \in D_\infty : s > q \}])$ *satisfies* $c(s) = c_\infty(s)$ *for all* $s \in D_\infty$ (where $c_\infty$ and $D_\infty$ are as in 6.6).
(2) *All* rational *Conway numbers (i. e. elements of* $c[\mathbb{Q}]$*) are located reals.*
(3) *$c$ is reflection preserving and strictly increasing, i. e. we have* $c(-q) \equiv -c(q)$ *and* $c(p) < c(q)$ *if* $p < q$ *and* $p, q \in \mathbb{Q}$.
(4) $\overline{c} := \overline{\mathrm{pr}} \circ c$ *defines an* injective group homomorphism $\overline{c} : \mathbb{Q} \longrightarrow \mathbf{No}_\star$ *with image* $\mathbf{Q} := \overline{c}[\mathbb{Q}] \subset \mathbf{R}$ (where $\overline{\mathrm{pr}} : \mathrm{No}_\star \longrightarrow \mathbf{No}_\star$ is as in 6.6).
(5) *Each of the following statements is equivalent to* (LEM) *from* 1.9:
- (a) *Every $z \in \mathbf{R}$ can be approximated on both sides by rational numbers, i. e. for all $z \in \mathbf{R}$ and all $k \in \mathbb{N}$ there are $r_1, r_2 \in \mathbf{Q}$ with* $$z - 2^{-k} < r_1 < z < r_2 < z + 2^{-k}.$$
- (b) *All Conway reals are located.*

Proof: (1),(2): $c(q)$ is a located real Conway number because $c_\infty$ is strictly increasing, while $c(s) = c_\infty(s)$ can be seen with 5.4.
(3) is proved using the definition of $c$, and (4) is proved with 6.5.
(5): "(a) $\Longleftrightarrow$ (b)" is proved with 7.3, "(LEM) $\Longrightarrow$ (b)" follows from 1.9 (2), and for "(b) $\Longrightarrow$ (LEM)" use the real Conway number $z_\psi \equiv -x_\psi$ with $x_\psi$ as in the proof of 1.9. □

7.5 Definition *(Rational cuts)*
A *rational cut* is a pair $(P, Q)$ of subsets $P, Q \subseteq \mathbb{Q}$ such that
- (C1) $P$ and $Q$ are *downwards* resp. *upwards closed*: $\downarrow P = P$ and $\uparrow Q = Q$, i. e. $p < p' \in P \Longrightarrow p \in P$ resp. $q > q' \in Q \Longrightarrow q \in Q$;
- (C2) $P$ and $Q$ are *disjoint*, thus $P < Q$, i. e. $p < q$ for all $p \in P$, $q \in Q$;
- (C3) $P$ and $Q$ are *open*, i. e. for every $p \in P$ there is a $p' \in P$ with $p' > p$ and for every $q \in Q$ there is a $q' \in Q$ with $q' < q$.

A rational cut $(P, Q)$ is called *bounded* if $P$ and $Q$ are non-empty, and $(P, Q)$ is called *located* if $P \cup Q$ is dense in $\mathbb{Q}$ (equivalently $p \in P$ or $q \in Q$ whenever $p, q \in \mathbb{Q}$ with $p < q$).

7.6 Theorem (Real Conway numbers and rational cuts)
(1) *For every rational cut $(P, Q)$ there is a Conway number $\hat{c}(P, Q) \equiv (c[P], c[Q])$ satisfying* (R2)*, which is real (and located) if $(P, Q)$ is bounded (and located).*
(2) *For every real Conway number $z$ there is a rational cut $\check{c}(z) := (P_z, Q_z)$ with $P_z := \{ p \in \mathbb{Q} : c(p) < z \}$ and $Q_z := \{ q \in \mathbb{Q} : c(q) > z \}$, which is located if $z$ is located. Whenever $(P, Q)$ is located we have $\check{c}(\hat{c}(P, Q)) = (P, Q)$.*
(3) *There is a bijection between $\mathbf{R}$, defined in* 7.1 *to be the set of located Conway reals modulo $=$, and the set $\mathbb{R}$ of* Dedekind reals*, i. e. bounded and located rational cuts.*

Proof: (1) is proved straightforwardly.
(2): The assertions of the first sentence are verified easily, and $s < \hat{c}(P,Q)$ can hold with a located rational cut $(P,Q)$ only if $s \in P$.
(3): For located reals we have $\mathrm{L}_z \subseteq \downarrow c[P_z]$, $\mathrm{R}_z \subseteq \uparrow c[Q_z]$ using 7.3 (1), thus $z = \hat{c}(\check{c}(z))$ holds by the Simplicity Lemma 6.5, as clearly $c[P_z] < z < c[Q_z]$. □

# Appendix
# On Ordinals

## A Ordinal Numbers

A.1 Motivation In [8] Georg Cantor constructed for any point set $P$ a sequence of derived sets $P', P'', \ldots$; in [9] he generalized this construction and obtained a sequence $P', P'', \ldots, P^{(\infty)}, P^{(\infty+1)}, P^{(\infty+2)}, \ldots$ containing derived sets of infinite orders. The generalized numbers appearing here as orders of derived sets were called *ordinal numbers* ("Ordnungszahlen") by Cantor. In [10] he defined sums and products for these numbers and wrote $\omega$ instead of $\infty$.

In the twentieth century other authors dealt with ordinal numbers as well-ordered sets, e. g. Luitzen E. J. Brouwer in [7] and John von Neumann in [20]. This approach is nowadays preferred and leads to a proper class On, the collection of all ordinal numbers, cf. e. g. [21]. For the Conway theory presented in this paper it is sufficient to use a set $\mathrm{On}_\star$ of ordinal numbers (defined in A.7) instead of the proper class On.

The following definition adapts the notion of ordinal numbers given by Per Martin-Löf in [18] p. 79ff and generalizes his definition of the second number class. (W-types are not used.)

A.2 Definition *(Ordinal numbers)*
For every natural number $j \in \mathbb{N}_0$ define recursively a set $\mathrm{On}_j$ satisfying the following conditions. (The set $\mathrm{On}_j$ may be called *j-th ordinal number class.*)
(i) $0 \in \mathrm{On}_j$;
(ii) for any $i \in \{0, \ldots, j-1\}$ and every function $l : \mathrm{On}_i \longrightarrow \mathrm{On}_j$ there is an element $\mathrm{Suc}_i^j(l)$ in $\mathrm{On}_j$;
(iii) every element of $\mathrm{On}_j$ is constructed by (i) and (ii) in a finite number of steps.
(For the functions $l$ in (ii) *extensionality*, i. e. $\alpha = \beta \Longrightarrow l(\alpha) = l(\beta)$, is out of question, because up to now we have not defined any equality relation on $\mathrm{On}_j$; such a definition will be given in B.2.
Some readers might like to replace "function" by "operation" in (ii) as well as in the whole appendix, understanding an *operation* to be a "nonextensional function", cf. [5] p. 15. Here the notion of operation is taken to be primitive, it cannot be reduced to that of an extensional function with multiple values as proposed in [19] p. 30f and included in [6] p. 54.
Similarly, it is possible to put "preset" instead of "set" throughout the appendix, understanding a *preset* to be a "set without equality", cf. [3] p. 34f.)

A.3 Hint
We may say that elements of the form $\mathrm{Suc}_i^j(l)$ *contain* each $l(\alpha)$ ($\alpha \in \mathrm{On}_i$, $l : \mathrm{On}_i \longrightarrow \mathrm{On}_j$, $i \in \{0, \ldots, j-1\}$, $j \in \mathbb{N}_0$).

$\mathrm{On}_j$ satisfies a *Descending Chain Condition* with respect to this relation of containment (denoted by $\prec$): Because of A.2 (iii) there is no infinite sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in \mathrm{On}_j$ and $\alpha_{n+1} \prec \alpha_n$ for all $n \in \mathbb{N}$.

A.4 Observation

The following *Principle of Transfinite Induction* holds due to the inductive definition of the sets $\mathrm{On}_j$.

To prove a proposition $\psi(\alpha)$ for all $\alpha \in \mathrm{On}_j$ we must prove the Induction Basis $\psi(0)$, and for each function $l : \mathrm{On}_i \longrightarrow \mathrm{On}_j$ $(i \in \{0, \ldots, j-1\})$ we have to prove the Induction Step $\big(\forall\, \alpha \in \mathrm{On}_i : \psi(l(\alpha))\big) \Longrightarrow \psi(\mathrm{Suc}_i^j(l))$.

We also have a form of *Definition by Transfinite Recursion* for functions with ordinal numbers as arguments.

To define a function $f : \mathrm{On}_j \longrightarrow X$ from $\mathrm{On}_j$ to a set $X$, we must define $f(0) \in X$, and for each function $l : \mathrm{On}_i \longrightarrow \mathrm{On}_j$ $(i \in \{0, \ldots, j-1\})$ we have to define $f(\mathrm{Suc}_i^j(l)) \in X$, possibly using already defined values $f(l(\alpha)) \in X$ $(\alpha \in \mathrm{On}_i)$.

A.5 Remark *(Recursion operators)*

For $\mathbb{N}_0$ there is a *natural recursion operator* "rec" with $\mathrm{rec}(x, f) : \mathbb{N}_0 \longrightarrow X$ for any set $X$, $x \in X$ and $f : \mathbb{N}_0 \times X \longrightarrow X$, such that $\mathrm{rec}(x, f)(0) = x$ and $\mathrm{rec}(x, f)(n+1) = f(n, \mathrm{rec}(x, f)(n))$ for every $n \in \mathbb{N}_0$. The corresponding recursion operator for $\mathrm{On}_1$ is "$\mathrm{rec}_1$" with $\mathrm{rec}_1(x_0, f_0) : \mathrm{On}_1 \longrightarrow X$ for any set $X$, $x_0 \in X$ and $f_0 : {\mathrm{On}_1}^{\mathrm{On}_0} \times X^{\mathrm{On}_0} \longrightarrow X$, such that $\mathrm{rec}_1(x_0, f_0)(0) = x_0$ and $\mathrm{rec}_1(x_0, f_0)(\mathrm{Suc}_0^1(l_0)) = f_0(l_0, \mathrm{rec}_1(x_0, f_0) \circ l_0)$ for every $l_0 : \mathrm{On}_0 \longrightarrow \mathrm{On}_1$. Similarly, the *recursion operator* for $\mathrm{On}_j$ is "$\mathrm{rec}_j$" $(j \in \mathbb{N}_0)$ with

$$\mathrm{rec}_j(x_0, f_0, f_1, \ldots, f_{j-1}) : \mathrm{On}_j \longrightarrow X \text{ for any set } X,\ x_0 \in X,$$
$$f_i : {\mathrm{On}_j}^{\mathrm{On}_i} \times X^{\mathrm{On}_i} \longrightarrow X \ (i \in \{0, \ldots, j-1\}),$$
$$\text{such that}$$
$$\mathrm{rec}_j(x_0, f_0, f_1, \ldots, f_{j-1})(0) = x_0 \text{ and}$$
$$\mathrm{rec}_j(x_0, f_0, f_1, \ldots, f_{j-1})(\mathrm{Suc}_i^j(l_i)) = f_i(l_i, \mathrm{rec}_j(x_0, f_0, f_1, \ldots, f_{j-1}) \circ l_i)$$
$$\text{for every } l_i : \mathrm{On}_i \longrightarrow \mathrm{On}_j \ (i \in \{0, \ldots, j-1\}).$$

A.6 Note

For any $j, j' \in \mathbb{N}_0$ with $j < j'$ we obtain by transfinite recursion (cf. A.4) a function $h_{j\,j'} : \mathrm{On}_j \longrightarrow \mathrm{On}_{j'}$, with $0 \longmapsto 0$ and $\mathrm{Suc}_i^j(l) \longmapsto \mathrm{Suc}_i^{j'}(h_{j\,j'} \circ l)$ for every $l : \mathrm{On}_i \longrightarrow \mathrm{On}_j$, $i < j$.

Similarly we obtain functions $g_{j'j} : \mathrm{On}_{j'} \longrightarrow \mathrm{On}_j$ $(j, j' \in \mathbb{N}_0,\ j < j')$ with $0 \longmapsto 0$, $\mathrm{Suc}_i^{j'}(l) \longmapsto \mathrm{Suc}_i^j(g_{j'j} \circ l)$ whenever $l : \mathrm{On}_i \longrightarrow \mathrm{On}_{j'}$, $i \in \{0, \ldots, j-1\}$, and $\mathrm{Suc}_i^{j'}(l) \longmapsto g_{j'j}(l(0))$ whenever $l : \mathrm{On}_i \longrightarrow \mathrm{On}_{j'}$, $i \in \{j, \ldots, j'-1\}$. (With respect to $=$ as in B.2 these functions are *not extensional* in the sense of A.2.)

Because $g_{j'j}(h_{j\,j'}(\alpha))$ is $\alpha$ for all $\alpha \in \mathrm{On}_j$ (by transfinite recursion, cf. A.4), $h_{j\,j'}$ is an embedding of $\mathrm{On}_j$ into $\mathrm{On}_{j'}$ for every $j < j'$.

A.7 Convention

By dint of the functions $h_{j\,j'}$ from A.6 we can interpret elements of $\mathrm{On}_j$ as elements of any $\mathrm{On}_{j'}$ with $j' > j$, and we will write $\mathrm{Suc}_i$ instead of $\mathrm{Suc}_i^{i+1}$ as well as instead of $\mathrm{Suc}_i^j$ whenever the choice of $j > i$ does not matter. Furthermore we define $\mathrm{On}_\star := \bigcup_{j=0}^{\infty} \mathrm{On}_j$ ; elements of this set may be called *ordinal numbers*.

A.8 Notation
Let $\mathrm{l}_\alpha : \mathrm{On}_0 \longrightarrow \mathrm{On}_\star,\ x \longmapsto \alpha$ denote the constant function with value $\alpha \in \mathrm{On}_\star$. Then we have a *successor function* $\mathrm{suc} : \mathrm{On}_\star \longrightarrow \mathrm{On}_\star,\ \alpha \longmapsto \mathrm{Suc}_0(\mathrm{l}_\alpha)$.

It is customary to distinguish three *forms of ordinal numbers*:

0. the *zero ordinal* 0,
1. *successor ordinals* of the form $\mathrm{suc}(\alpha)$ with $\alpha \in \mathrm{On}_\star$,
2. lim-*ordinals* of the form $\mathrm{Suc}_i(l)$ with $l : \mathrm{On}_i \longrightarrow \mathrm{On}_j$, $0 < i < j$.

For lim-ordinals $\lambda$ it is sometimes convenient to use a notation like $\lim_{\alpha \in \mathrm{On}_i} l(\alpha)$ instead of $\mathrm{Suc}_i(l)$. Such a $\lambda$ would be called *limit ordinal* if $l$ is strictly increasing with respect to the relation $<$ defined in B.2.

A.9 Examples
(0) The zero ordinal 0 is the only element of $\mathrm{On}_0$.

(1) The elements of $\mathrm{On}_\star$ constructed from 0 by applying the function suc a finite number of times may be called *finite* ordinal numbers. Let us denote suc(0) by 1, suc(1) by 2, suc(2) by 3, etc. Then we see that $\mathrm{On}_1$, the set of all finite ordinal numbers, is just a disguised form of the set $\mathbb{N}_0$ of natural numbers. (Even though $\mathrm{On}_1$ has infinitely many elements, the set $\mathrm{On}_1$ is finitely presented.)

(2) Because of (1), we can identify $\mathrm{On}_1$-sequences with ordinary sequences. Let $\omega$ denote $\mathrm{Suc}_1(0, 1, 2, \ldots)$, i. e. $\omega$ is $\mathrm{Suc}_1(h_{12}) \in \mathrm{On}_2$ with $h_{12}$ from A.6. (As $h_{12}$ is $\mathrm{rec}_1(0, f_0)$ with $f_0(l_0, l) := \mathrm{Suc}_1^2(l)$ for $l_0 : \mathrm{On}_0 \longrightarrow \mathrm{On}_1$, $l : \mathrm{On}_0 \longrightarrow \mathrm{On}_2$ and $\mathrm{rec}_1$ as in A.5, the construction of $\omega$ requires only a finite number of steps.) Then we have a transfinite sequence $0, 1, 2, \ldots, \omega, \mathrm{suc}(\omega), \mathrm{suc}(\mathrm{suc}(\omega)), \ldots$ similar to Cantor's sequence of derivational orders mentioned in A.1. By means of B.5 we will be able to write $\omega + 1$ for $\mathrm{suc}(\omega)$, $\omega + 2$ for $\mathrm{suc}(\mathrm{suc}(\omega))$, etc.

(3) More generally, let $\omega_{j-1}$ denote the ordinal number $\mathrm{Suc}_j(h_{j\,j+1})$ for every $j \in \mathbb{N}$ (with $h_{j\,j+1}$ as defined in A.6). Then $\omega_0$ is $\omega$, and for any $j \in \mathbb{N}$ the element $\omega_{j-1}$ in $\mathrm{On}_{j+1}$ contains every element of $\mathrm{On}_j$, i. e. $\alpha \prec \omega_{j-1}$ holds for all $\alpha \in \mathrm{On}_j$ (if $h_{j\,j+1}(\alpha)$ is identified with $\alpha$ as in A.7; for $\prec$ cf. A.3).

# B Ordinal order and ordinal addition

B.1 Motivation
There is a function $c_\star : \mathrm{On}_\star \longrightarrow \mathrm{No}_\star$ which meets the following conditions.
(i) $c_\star(0) \equiv \{\,|\,\} \equiv \bot$, *the neutral element of* $\mathrm{No}_\star$,
(ii) $c_\star(\mathrm{suc}(\alpha)) \equiv \{c_\star(\alpha)|\}$ *for all* $\alpha \in \mathrm{On}_\star$,
(iii) $c_\star(\lambda) \equiv (c_\star[l[\mathrm{On}_i]], \emptyset)$ *for all* lim-*ordinals* $\lambda$ *of the form* $\mathrm{Suc}_i(l)$ *with* $l : \mathrm{On}_i \longrightarrow \mathrm{On}_j,\ 0 < i < j$.
(Construct by transfinite recursion (cf. A.4) functions $c_j : \mathrm{On}_j \longrightarrow \mathrm{No}_\star$, $j \in \mathbb{N}_0$, satisfying $c_j[\mathrm{On}_j] \subset \mathrm{No}_j$ as well as (i), (ii) and (iii) with $c_j$ instead of $c_\star$. Then define $c_\star(\alpha)$ to be $c_j(\alpha)$ if $\alpha \in \mathrm{On}_j$.)

By transfinite induction (cf. A.4) we have $c_\star(\alpha) \in N_\alpha$ for all $\alpha \in \mathrm{On}_\star$, so $c_\star[\mathrm{On}_\star] \subset \mathrm{No}_\star$. Conway numbers of the form $c_\star(\alpha)$ with $\alpha \in \mathrm{On}_\star$ may be called *Conway ordinals*.

As $\mathrm{L}_{c_\star(\alpha)} = c_\star[\{\,\alpha' \in \mathrm{On}_\star : \alpha' \prec \alpha\,\}]$ (for $\prec$ see A.3) and $\mathrm{R}_{c_\star(\alpha)} = \emptyset$ for all $\alpha \in \mathrm{On}_\star$, characterization 4.3 (1),(2) might motivate Definition B.2. (Because of 6.4 (2) we write $<$ instead of $\lhd$.)

B.2 DEFINITION *(Ordinal order)*
For ordinal numbers $\alpha, \beta \in \mathrm{On}_\star$ define

$$\alpha \le \beta \;:\Longleftrightarrow\; \forall\, \alpha' \prec \alpha : \alpha' < \beta, \qquad \alpha \ge \beta \;:\Longleftrightarrow\; \beta \le \alpha,$$
$$\alpha < \beta \;:\Longleftrightarrow\; \exists\, \beta' \prec \beta : \alpha \le \beta', \qquad \alpha > \beta \;:\Longleftrightarrow\; \beta < \alpha,$$
$$\alpha = \beta \;:\Longleftrightarrow\; \alpha \le \beta \text{ and } \beta \le \alpha.$$

B.3 NOTE *For all ordinal numbers $\alpha, \beta \in \mathrm{On}_\star$ we have*
(1) $0 \le \alpha$ and $\neg(\alpha < 0)$,
(2) $\neg(\alpha \le \beta$ and $\alpha > \beta)$,
(3) $\alpha \le \alpha$,
(4) $\alpha \prec \beta \Longrightarrow \alpha < \beta$.
(0 does not contain any ordinal number, so (1) is plain; (2) and (3) are easily proved by transfinite induction, and (4) is a consequence of (3).)

B.4 LEMMA (Properties of ordinal order)
*For all ordinal numbers $\alpha, \beta, \gamma \in \mathrm{On}_\star$ the following statements hold.*
(1) $\alpha \le \beta$ and $\beta \le \gamma \Longrightarrow \alpha \le \gamma$,
(2) $\alpha \le \beta$ and $\beta < \gamma \Longrightarrow \alpha < \gamma$,
(3) $\alpha < \beta$ and $\beta \le \gamma \Longrightarrow \alpha < \gamma$,
(4) $\alpha < \beta$ and $\beta < \gamma \Longrightarrow \alpha < \gamma$,
(5) $\alpha < \beta \Longrightarrow \alpha \le \beta$.

PROOF:
(1), (2) and (3) are mutually proved by transfinite induction:
1) $\alpha' \prec \alpha \le \beta \le \gamma \Longrightarrow \alpha' < \beta \le \gamma$ [by B.3 (4) and Ind. Hyp. (3) ]
$\Longrightarrow \alpha' < \gamma$ [by Ind. Hyp. (3)].
2) $\alpha \le \beta \le \gamma' \prec \gamma \Longrightarrow \alpha \le \gamma'$ [by Ind. Hyp. (1)].
3) $\alpha \le \beta' \prec \beta \Longrightarrow \alpha \le \beta' < \gamma$ [because $\beta \le \gamma$]
$\Longrightarrow \alpha < \gamma$ [by Ind. Hyp. (2)].
(4) and (5) are mutually proved by transfinite induction:
4) $\alpha < \beta \le \gamma' \prec \gamma \Longrightarrow \alpha < \gamma'$ [by (3)]
$\Longrightarrow \alpha \le \gamma'$ [by Ind. Hyp. (5)].
5) $\alpha' \prec \alpha < \beta \Longrightarrow \alpha' < \beta$ [by B.3 (4) and Ind. Hyp. (4)]. □

B.5 DEFINITION *(Ordinal addition)*
By transfinite recursion (cf. A.4) we define the *sum of two ordinal numbers* according to the form of the second summand (cf. e. g. Proposition 6.3.3 in [21])

$$\alpha + 0 := \alpha,$$
$$\alpha + \mathrm{suc}(\beta) := \mathrm{suc}(\alpha + \beta) \text{ for every } \beta \in \mathrm{On}_\star,$$
$$\alpha + \mathrm{Suc}_i(l) := \lim_{\gamma \in \mathrm{On}_i}(\alpha + l(\gamma)) \text{ whenever } l : \mathrm{On}_i \longrightarrow \mathrm{On}_j,\ 0 < i < j.$$

So we obtain a function $\mathrm{add}_{\mathrm{On}_\star} : \mathrm{On}_\star \times \mathrm{On}_\star \longrightarrow \mathrm{On}_\star,\ (x, y) \longmapsto x + y$ satisfying $\mathrm{add}_{\mathrm{On}_\star}(\alpha, \beta) \in \mathrm{On}_j$ whenever $\alpha, \beta \in \mathrm{On}_j$.

B.6 EXAMPLES
(1) Ordinal addition on $\mathrm{On}_1$ can be seen to be just the ordinary addition for natural numbers (cf. A.9 (1)).
(2) We have $\alpha < \alpha + 1$ for all $\alpha \in \mathrm{On}_\star$ and $\nu < \omega$ for all $\nu \in \mathrm{On}_1$:

$$0 < 1 < 2 < \ldots < \omega < \omega + 1 < \omega + 2 < \ldots \quad \text{(cf. A.9)}.$$

(3) Because of $1 + \omega = \omega < \omega + 1$, ordinal sums generally depend on the order of summation.

B.7 Proposition *(Properties of ordinal addition)*
*For all ordinal numbers $\alpha, \beta, \gamma \in \mathrm{On}_\star$ the following statements hold.*
(1) $0 + \gamma = \gamma$,
(2) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
(3) $\alpha \leq \beta \implies \alpha + \gamma \leq \beta + \gamma$,
(4) $\alpha = \beta \implies \alpha + \gamma = \beta + \gamma$,
(5) $\alpha + \beta \leq \alpha + \gamma \iff \beta \leq \gamma$,
(6) $\alpha + \beta < \alpha + \gamma \iff \beta < \gamma$,
(7) $\alpha + \beta = \alpha + \gamma \iff \beta = \gamma$.
(Associativity of ordinal addition is well known since Cantor, cf. [10] p. 550f. As $0 + \omega = 1 + \omega$, (3) with $<$ instead of $\leq$ does not hold, and (5), (6) and (7) with reversed order of summation are not valid.)

Proof:
(1), (2) and (3) are proved straightforwardly by transfinite inductions (on $\gamma$), (4) is a consequence of (3); (5) and (6) are mutually proved by transfinite induction, and (7) is a consequence of (5). □

B.8 Result ($\mathrm{On}_\star$ monoid)
$\leq$ is a *preorder relation* (reflexive and transitive) on $\mathrm{On}_\star$ with associated equivalence relation $=$, and $<$ is a strict partial order relation on $\mathrm{On}_\star$. Thus $\mathbf{On}_\star$ (i. e. $\mathrm{On}_\star$ modulo $=$) with ordinal addition and ordinal order is an ordered monoid, having 0 as neutral element. In $\mathbf{On}_\star$ the left cancellation law does hold, but $\mathbf{On}_\star$ is not commutative.

B.9 Remarks
(1) The function $c_\star$ from B.1 is extensional (in the sense of A.2), injective and strictly increasing: We have $c_\star(\alpha)\, \varrho\, c_\star(\beta) \iff \alpha\, \varrho\, \beta$ for all $\alpha, \beta \in \mathrm{On}_\star$ and for every $\varrho \in \{\leq, <, =\}$. Furthermore $c_\star$ satisfies $c_\star(\alpha + \beta) \leq c_\star(\alpha) + c_\star(\beta)$ for all $\alpha, \beta \in \mathrm{On}_\star$; $c_\star$ is *not a homomorphism*, as this inequality is strict for $\alpha = 1$, $\beta = \omega$.
(2) A necessary and sufficient condition for $\mathrm{On}_{j+1}$ to be *totally ordered*, i. e. $\alpha \leq \beta$ or $\alpha > \beta$ for all $\alpha, \beta \in \mathrm{On}_{j+1}$, is the *$j$-th limited principle of omniscience* ($\mathrm{LPO}_j$) $\forall\, \alpha \in \mathrm{On}_j : l(\alpha) = 0$ or $\exists\, \alpha \in \mathrm{On}_j : l(\alpha) = 1$ for each $l : \mathrm{On}_j \longrightarrow \{0, 1\}$. (If $\mathrm{On}_1$ is replaced by the set of natural numbers in ($\mathrm{LPO}_1$) we obtain Bishop's LPO as in [5] p. 3 or [19] p. 4. Because (LEM) implies ($\mathrm{LPO}_j$) for any $j \in \mathbb{N}_0$, in classical mathematics $\mathrm{On}_\star$ is totally ordered.)

First observe that ($\mathrm{LPO}_j$) implies ($\mathrm{LPO}_i$) for every $i \in \{0, \ldots, j-1\}$:
For $l : \mathrm{On}_i \longrightarrow \{0, 1\}$ apply ($\mathrm{LPO}_j$) to $l \circ g_{j\,i}$ with $g_{j\,i}$ as in A.6.

Now prove sufficiency by transfinite induction on $\alpha$:
For $\alpha = \mathrm{Suc}_i^{j+1}(l)$ with $l : \mathrm{On}_i \longrightarrow \mathrm{On}_{j+1}$ and $\beta \in \mathrm{On}_{j+1}$ define $l_\beta : \mathrm{On}_i \longrightarrow \{0, 1\}$ with $l_\beta(\gamma) = 0$ if $l(\gamma) < \beta$ and $l_\beta(\gamma) = 1$ if $l(\gamma) \geq \beta$ (this is possible by Ind. Hyp.). Apply ($\mathrm{LPO}_i$) to $l_\beta$, then use $\alpha \leq \beta \iff \forall\, \gamma \in \mathrm{On}_i : l_\beta(\gamma) = 0$ and $\alpha > \beta \iff \exists\, \gamma \in \mathrm{On}_i : l_\beta(\gamma) = 1$.

($\mathrm{LPO}_j$) is necessary because for every $l : \mathrm{On}_j \longrightarrow \mathrm{On}_{j+1}$ with $l[\mathrm{On}_j] \subseteq \{0, 1\}$ we have $\mathrm{Suc}_j(l) \leq 1 \iff \forall\, \alpha \in \mathrm{On}_j : l(\alpha) < 1 \iff \forall\, \alpha \in \mathrm{On}_j : l(\alpha) = 0$, and $\mathrm{Suc}_j(l) > 1 \iff \exists\, \alpha \in \mathrm{On}_j : l(\alpha) \geq 1 \iff \exists\, \alpha \in \mathrm{On}_j : l(\alpha) = 1$.

(3) It is possible to define *ordinal multiplication* by transfinite recursion in a manner similar to B.5 (cf. e. g. Proposition 6.4.3 in [21]).

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