The Serre-Rost Invariant of Albert Algebras in Characteristic three.

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0. Introduction. The authors [5] have recently developed an elementary approach to the Serre-Rost invariant of Albert algebras that is valid in all characteristics except 3. In this special case, Serre [9] has defined the invariant in a different way and established its existence by using Rost's original results [6] in characteristic zero and reducing them mod 3. It is the purpose of the present note to show that the elementary approach of [5] survives in characteristic 3 as well once the necessary modifications of the cohomological set-up as indicated in [8] have been carried out.

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1. The reader is assumed to be familiar with the terminology, notations and results of [5]. We fix an arbitrary base field k of characteristic $p > 0$ and write $\Omega = \Omega_k := \Omega_{k/\mathbf{Z}}$ for the absolute universal differential algebra of k [2]. As a graded k-algebra, $\Omega = \bigoplus_{q\geq 0} \Omega^q$ is just the exterior algebra of $\Omega^1 = \Omega^1_{k/\mathbf{Z}}$, the vector space of Kähler differentials of k over the integers; also, Ω comes

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equipped with a universal differentiation, which is an additive map $d : \Omega \to \Omega$ of degree 1.

2. Setting $\Omega^q = 0$ for $q < 0$, we follow [8, 10.1] to recall that, for each $q \geq 1$, there is a natural p-linear map

$$\gamma : \Omega^{q-1} \longrightarrow \Omega^{q-1}/d\Omega^{q-2}$$

satisfying

$$\gamma(u\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_{q-1}}{x_{q-1}}) = u^p \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_{q-1}}{x_{q-1}} \bmod d\Omega^{q-2}$$

for $u \in k,\ x_1, \ldots, x_{q-1} \in k^\times$. According to Kato [1] and Milne [3], the group

$$H_p^q(k) := \operatorname{coker}(\gamma - \pi),$$

π being the canonical projection $\Omega^{q-1} \to \Omega^{q-1}/d\Omega^{q-2}$, is the analogue in characteristic p of the groups $H^q(k, \boldsymbol{\mu}_p^{\otimes q-1})$ in characteristic $\neq p$. Observe that there is a natural epimorphism

$$\Omega^{q-1} \longrightarrow H_p^q(k),\ \omega \longmapsto <\omega>,$$

whose kernel is spanned by $d\Omega^{q-2}$ and the elements

$$(u^p - u)\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_{q-1}}{x_{q-1}} \qquad\qquad (u \in k, x_1, \ldots, x_{q-1} \in k^\times).$$

We have

$$H_p^1(k) = k/\wp k = H^1(k, \mathbf{Z}/p\mathbf{Z}),$$

where $\wp$ is the Artin-Schreier map $u \mapsto u^p - u$.

3. The groups $H_p^q(k), q \geq 1$, are clearly functorial in k, so for every field extension l/k we have a natural map, which we call *restriction*,

$$\operatorname{res}_{l/k} : H_p^q(k) \longrightarrow H_p^q(l).$$

Conversely, if l/k is separable of finite degree, we may identify $\Omega_l = \Omega_k \otimes_k l$ canonically, and the trace form of l/k yields a map in the opposite direction, which we call *corestriction*,

$$\operatorname{cor}_{l/k} : H_p^q(l) \longrightarrow H_p^q(k),$$

such that

$$\mathrm{cor}_{l/k} \circ \mathrm{res}_{l/k} = [l : k]\mathbf{1}.$$

In particular, as in classical Galois cohomology, $\mathrm{res}_{l/k}$ *is injective unless* p *divides* $[l : k]$. Finally, if l/k is a finite Galois extension with Galois group G, we conclude

$$\mathrm{res}_{l/k} \circ \mathrm{cor}_{l/k} = \sum_{\sigma \in G} \sigma^*,$$

where σ^* denotes the natural action of $\sigma \in G$ on $H_p^q(l)$.

4. The natural map from $k^\times$ to $k^\times/k^{\times p}$ will be symbolized by $a \mapsto < a >$. (This should not be confused with the map $a \mapsto [a]$ from $k^\times$ to $H^1(k, \boldsymbol{\mu}_n)$ for n not divisible by p, cf. [5, 1.6].) It is straightforward to check that there is a unique $\mathbf{Z}$-bilinear map

$$H_p^q(k) \times (k^\times/k^{\times p}) \longrightarrow H_p^{q+1}(k)$$

satisfying

$$(< \omega >, < a >) \longmapsto < \omega > \cdot < a > = [\omega, a) \ = < \omega \wedge \frac{da}{a} >$$

for $\omega \in \Omega^{q-1}$, $a \in k^\times$ (See [7], Chap XIV, §5 for similar expressions). This map serves as a substitute for the cup product in cohomology. In particular, it is stable under base change, and expressions like $(< \omega > \cdot < a >) \cdot < b >$ are alternating in $< a >, < b > \in k^\times/k^{\times p}$.

5. Let E/k be a cyclic field extension of degree p and σ a generator of its Galois group. Then some $y \in E$ has ${}^\sigma y = y + 1$, forcing $x = y^p - y \in k$, and $[E, \sigma] = < x >$ in $H^1(k, \mathbf{Z}/p\mathbf{Z}) = H_p^1(k)$. On the other hand, ${}_p\mathrm{Br}(k)$, the p-torsion part of the Brauer group, identifies with $H_p^2(k)$ in such a way that, if $D = (E/k, \sigma, c)$ is a cyclic algebra of degree p over k, we have

(5.1) $$[D] = [E, \sigma] \cdot < c >$$

for the corresponding element in $H_p^2(k)$; see [11] for details.

6. Recall that if D is a central simple associative algebra of degree 3 over k and $a \in k^\times$ then $V = D_0 \oplus D_1 \oplus D_2$, where $D_i = D$ $(0 \le i \le 2)$, can be endowed with a quadratic Jordan algebra structure induced by the cubic norm

$$N(x) := N_D(x_0) + aN_D(x_1) + a^{-1}N_D(x_2) - T_D(x_0x_1x_2)$$

for $x = (x_0, x_1, x_2) \in D$ and the base point $1 = (1_D, 0, 0), 1_D, N_D, T_D$ being the unit element, reduced norm, reduced trace, respectively, of D. By this, the *first Tits construction*, we obtain an Albert algebra, written as $\mathcal{J}(D, a)$. Conversely, given any Albert algebra $\mathcal{J}$ over k, then either $\mathcal{J} \cong \mathcal{J}(D, a)$ as above or there exists a quadratic field extension K/k such that $\mathcal{J} \otimes_k K \cong \mathcal{J}(D, a)$ over K.
We can now state the main result of the paper.

7. Theorem. (Serre [9]). *Let k be a field of characteristic 3. Then there exists a unique invariant assigning to each Albert algebra $\mathcal{J}$ over k an element*

$$g_3(\mathcal{J}) \in H^3_3(k)$$

which only depends on the isomorphism class of $\mathcal{J}$ and satisfies the following two conditions.

SR1 *If $\mathcal{J} \cong \mathcal{J}(D, a)$ for some central simple associative algebra D of degree 3 over k and some $a \in k^\times$ is a first Tits construction, then*

$$g_3(\mathcal{J}) = [D] \cdot < a > \in H^3_3(k).$$

SR2 *g_3 is invariant under base change, i.e.,*

$$g_3(\mathcal{J} \otimes_k l) = \mathrm{res}_{l/k}(g_3(\mathcal{J}))$$

for any field extension l/k.

Moreover, we have

SR3 *g_3 characterizes Albert division algebras, i.e., $\mathcal{J}$ is a division algebra iff $g_3(\mathcal{J}) \neq 0$.*

8. We first prove existence and uniqueness of the invariant g_3. To do so, we briefly summarize the contents of sections 3 and 4 in [5] and indicate the minor changes nesessary in characteristic 3.

If we define

$$g_3(\mathcal{J}) := [D] \cdot < a > \in H^3_3(k) \tag{8.1}$$

for a first Tits construction $\mathcal{J} \cong \mathcal{J}(D, a)$ as in 6. and

$$g_3(\mathcal{J}) := -\mathrm{cor}_{K/k}(g_3(\mathcal{J} \otimes_k K)) \tag{8.2}$$

for any separable quadratic field extension K/k such that $\mathcal{J} \otimes_k K$ is a first Tits construction then, as in [5, 3.4 - 3.7], one can show that the invariant is unique and that (8.2) is well defined provided (8.1) is.

Next we need the characteristic-3-version of [5, 4.3]:

9. Lemma. *Assume* $\mathrm{char}\, k = 3$, *let* D *be a central simple associative* k-*Algebra of degree* 3 *and* $b \in N_D(D^\times)$. *Then*

$$[D] \cdot < b > = 0.$$

Proof. We may assume that D is a division algebra and, by Zariski density, choose $u, y \in D^\times$ satisfying $N_D(u) = b,\ T_D(y^{-1}) \neq 0 \neq T_D(yu)$. This implies $b = N_D(y^{-1})N_D(yu)$, so by virtue of bilinearity (4.) we are allowed to assume that u generates an étale subalgebra of rank 3 in D. But then the proof may be completed in exactly the same manner as the one of [5, 4.3].

10. We now return to 8. and prove that (8.1) is well defined by considering a first Tits construction $\mathcal{J}$ as in 6, which, because of Lemma 9, may assumed to be a division algebra. Choosing internally a Jordan subalgebra $A \subset \mathcal{J}$, a central associative division algebra D of degree 3 over k, an isomorphism $\eta : D^+ \xrightarrow{\sim} A$ and an element $x \in \mathcal{J}$ which is associated with (D, η) [5, 4.4], it follows as in the proof of [5, 4.8] that

$$g_3(\mathcal{J}, A) := [D] \cdot < N_{\mathcal{J}}(x) >$$

depends only on A. (Observe that Lemma 9 takes care of the restriction on the characteristic in [5, 4.8], thereby removing the characteristic restrictions from [5, 4.12c)] and [5, 4.14] also.) It remains to show $g_3(\mathcal{J}, A) = g_3(\mathcal{J}, A')$ for any subalgebra $A' \subset \mathcal{J}$ having the form $A' \cong D'^+$ for some central simple associative k-algebra D' of degree 3. To this end we may assume that A, A'

contain a common cyclic cubic subfield [5, 4.9] (the proof of this result works also in characteristic 3 since we are allowed to start with arbitrary *separable* subfields). Then [5, 4.16] produces a chain of neighbors connecting A with A', and [5, 4.14] shows $g_3(\mathcal{J}, A) = g_3(\mathcal{J}, A')$, as claimed.

11. In view of 8. and 10., the only part of Theorem 7 demanding clarification is SR3. The easy direction follows from Lemma 9, so it remains to show that, conversely, $\mathcal{J}$ being a division algebra implies $g_3(\mathcal{J}) \neq 0$. To do so, we will follow Serre's argument in [9]. Let k be a field of characteristic $p > 0$. By results of Teichmüller [10], there is a local field K_0 of characteristic zero having residue field $\overline{K_0} = k$ and the property that $v_0(p) = 1$ where $v_0 : K_0^\times \to \mathbf{Z}$ is the valuation of K_0. Letting ζ be a primitive p-th root of unity, $K = K_0(\zeta)$ is totally ramified of degree $p-1$ over K_0 having $b = \zeta - 1$ as a local parameter. Write $\mathfrak{o}_K$ for the valuation ring of K and $u \mapsto \overline{u}$ for the natural map $\mathfrak{o}_K \to k$. Among the various results of Kato [1] relating the Galois cohomology of local fields to their residue fields, we only need the *existence* of natural homomorphisms (cf. [1, Theorem 2 (i)])

$$\kappa_p^q : H_p^q(k) \longrightarrow H^q(K, \mathbf{Z}/p\mathbf{Z})$$

satisfying

$$\kappa_p^1(<\overline{u}>) = [1 + b^p u] \qquad (u \in \mathfrak{o}_K), \tag{11.1}$$

$$\kappa_p^{q+1}(<\omega>\cdot<\overline{a}>) = \kappa_p^q(<\omega>) \cup [a] \quad (q \geq 1, \omega \in \Omega_k^{q-1}, a \in \mathfrak{o}_K^\times). \tag{11.2}$$

12. Keeping the situation described in 11., let E/K be an unramified cyclic field extension of degree p and σ a generator of its Galois group. Then one finds elements $x \in \mathfrak{o}_K^\times, y \in \mathfrak{o}_E^\times - K$ satisfying

$$(1 + by)^p = 1 + b^p x \ , \ {}^{\overline{\sigma}}\overline{y} = \overline{y} + 1 \ , \ \overline{y}^p - \overline{y} = \overline{x}.$$

By 5. and (11.1), this implies $\kappa_p^1([\overline{E}, \overline{\sigma}]) = [E, \sigma]$, which in turn, combining (5.1) with (11.2), yields

$$\kappa_p^2([\overline{D}]) = [D] \tag{12.1}$$

for any unramified cyclic division algebra D of degree p over k.

13. Assuming char $k = 3$ again, we can now complete the proof of SR3 by showing $g_3(\mathcal{J}) \neq 0$ for any Albert division algebra $\mathcal{J}$ over k. As usual, we may assume that $\mathcal{J} \cong \mathcal{J}(D, a)$ is a first Tits construction as in SR1. Using 11. and 12., we let $\mathcal{J}_1$ be the unique unramified Albert division algebra over K having $\overline{\mathcal{J}_1} \cong \mathcal{J}$ [4, Theorem 2]. By [4, Proposition 4], $\mathcal{J}_1 \cong \mathcal{J}(D_1, a_1)$ where D_1 is the unique unramified associative division algebra of degree 3 over K having $\overline{D_1} \cong D$ and $a_1 \in \mathfrak{o}_K^\times$ satisfies $\overline{a_1} = a$. Now SR1, (5.1), (11.2), (12.1) and [5, 3.2 SR3], combined, show

$$\kappa_p^3(g_3(\mathcal{J})) = g_3(\mathcal{J}_1) \neq 0,$$

forcing $g_3(\mathcal{J}) \neq 0$, as claimed.

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