Exceptional simple Jordan algebras and Galois cohomology

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Introduction Exceptional simple Jordan algebras have been classified a long time ago. It is known since the work of Tits ([J]) and McCrimmon ([M2]) that they may all be obtained by one of the two Tits constructions. The most delicate part of the proof is to establish the following fundamental fact: Given a central simple exceptional Jordan algebra $\mathcal{J}$ containing a subalgebra of the form A^+, the Jordan algebra determined by a central simple associative algebra A of degree 3, there exists a nonzero scalar μ in the base field such that the identity transformation of A^+ extends to an isomorphism from $\mathcal{J}$ onto the first Tits construction $\mathcal{J}(A,\mu)$. In [J], [M2] but also in [Sp] and, more generally, in [PR], this is accomplished by a careful analysis of the way in which A^+ sits in $\mathcal{J}$. The aim of the present note is to give a different proof, reducing Jordan theory to a minimum and relying instead on elementary facts from Galois cohomology.

1. We begin by recalling from [M1] the basic features of the first Tits construction. Let k be a field, remaining fixed throughout this paper. For a central simple associative algebra A of degree 3 over k and a nonzero scalar $\mu_0 \in k$, we put

$$\mathcal{J}_0 = \mathcal{J}(A,\mu_0) = A \oplus A \oplus A$$

as a vector space over k and define a cubic form N (the *norm*), a nondegenerate symmetric bilinear form T (the *trace*) as well as a quadratic map $\sharp$ (the *adjoint*) on $\mathcal{J}_0$ according to the rules

$$\begin{aligned} N(x) &= N_A(a_0) + \mu_0 N_A(a_1) + \mu_0^{-1} N_A(a_2) - T_A(a_0\, a_1\, a_2), \\ T(x,y) &= T_A(a_0,b_0) + T_A(a_1,b_2) + T_A(a_2,b_1), \\ x^\sharp &= (a_0^\sharp - a_1a_2, \mu_0^{-1}a_2^\sharp - a_0a_1, \mu_0 a_1^\sharp - a_2a_0) \end{aligned}$$

for

$$x = (a_0,a_1,a_2),\ y = (b_0,b_1,b_2) \in \mathcal{J}_0,$$

the symbols $N_A, T_A, \sharp$ on the right referring to the (reduced) norm, trace, adjoint, respectively, of A. Then, setting $x \times y = (x+y)^\sharp - x^\sharp - y^\sharp$, $\mathcal{J}_0$ becomes a central simple exceptional Jordan algebra over k under the U-operator

$$U_x y = T(x,y)x - x^\sharp \times y$$

and the identity element $1 = (1_A, 0, 0)$. Note that A^+, via the first summand, identifies with a subalgebra of $\mathcal{J}_0$.

2. A linear bijection $\mathcal{J}_0 \to \mathcal{J}_0$ is an automorphism if and only if it preserves the norm and the identity element. For example, setting

$$\Gamma_A = \{v \in A : N_A(v) = 1\}$$

and picking $v \in \Gamma_A$, the rule

$$(a_0, a_1, a_2) \mapsto (a_0, a_1 v^{-1}, v\, a_2)$$

determines an automorphism $\iota(v)$ of $\mathcal{J}_0$ *fixing* A^+. On the other hand, the automorphism group of $(\mathcal{J}_0, A^+)$, i. e., the group of automorphism of $\mathcal{J}_0$ *stabilizing* A^+, will be denoted by Aut $(\mathcal{J}_0, A^+)$. Via restriction, we obtain a canonical homomorphismus $\rho :$ Aut $(\mathcal{J}_0, A^+) \to$ Aut A^+.

3. Proposition *The short sequence*

$$1 \to \Gamma_A \underset{\iota}{\to} \text{Aut } (\mathcal{J}_0, A^+) \underset{\rho}{\to} \text{Aut } A^+ \to 1$$

is exact.

Proof. ι is clearly injective and $\rho \circ \iota = \mathbf{1}$. Hence we have exactness on the left, and in order to prove exactness in the middle, it suffices to show that any automorphism η of $\mathcal{J}_0$ fixing A^+ has the form $\iota(v)$ for some $v \in \Gamma_A$. Since the elements x of the second (resp. third) summand of $\mathcal{J}_0$ may be characterized intrinsically by the condition that they are orthogonal to A^+ relative to T and satisfy $a \times (b \times x) = -(ab) \times x$ (resp. $-(ba) \times x$) for all $a, b \in A^+$ [M2, p. 308], both summands are stabilized by η, so there are linear bijections $\eta_i : A \to A$ $(i = 1, 2)$ having $(a_0, a_1, a_2 \in A)$. $\eta\big((a_0, a_1, a_2)\big) = \big(a_0, \eta_1(a_1), \eta_2(a_2)\big)$ Expanding $\eta\big(a_0 \times (0, a_1, a_2)\big)$ in two different ways shows $\eta_1(a_0 a_1) = a_0\, \eta_1(a_1)$,
$\eta_2(a_2\, a_0) = \eta(a_2) a_0$. Hence there are $v, w \in A^\times$ such that $\eta_1(a) = aw$, $\eta_2(a) = va$ for all $a \in A$. Comparing the first components of $\eta\big((0, 1_A, 1_A)^\sharp\big) = \eta\big((0, 1_A, 1_A)\big)^\sharp$ now yields $w = v^{-1}$, and since η preserves the norm we have $v \in \Gamma_A$, forcing $\eta = \iota(v)$, as claimed. It remains to prove exactness on the right, i.e., that ρ is surjective. So let $\varphi \in$ Aut A^+. Then φ is either an automorphism or an antiautomorphism of A. In the first case, the rule $(a_0, a_1, a_2) \mapsto (\varphi(a_0), \varphi(a_1), \varphi(a_2))$ extends φ to an automorphism of $(\mathcal{J}_0, A^+)$. In the second case, A is necessarily split, so some $u \in A^+$ has $N_A(u) = \mu_0^2$, and we obtain an extension of φ to an automorphism of $(\mathcal{J}_0, A^+)$ via $(a_0, a_1, a_2) \mapsto (\varphi(a_0), \varphi(a_2) u^{-1}, u\, \varphi(a_1))$.

□

4. We now fix a separable closure k_s of k, with absolute Galois group
$G = \text{Gal}(k_s/k)$. Scalar extensions from k to k_s will be indicated by a subscript "s".

Then we have two short exact sequences of G-groups, one coming from **3.**,

$$1 \to \Gamma_{A_s} \to \text{ Aut } (\mathcal{J}_{0s}, A_s^+) \to \text{ Aut } A_s^+ \to 1,$$

the other coming from the reduced norm of A,

$$1 \to \Gamma_{A_s} \to A_s^\times \to k_s^\times \to 1.$$

Passing to the "long" exact sequences of nonabelian cohomology ([S1] VII Proposition 1) and observing $H^1(k, A_s^\times) = 1$ ([K], 1.7 Ex. 1), we obtain the following diagram with exact rows and columns (in the category of pointed sets):

$$\begin{array}{ccccc}
 & A^\times & & & \\
 & \downarrow N_A & & & \\
 & k^\times & & & \\
 & \downarrow \delta & & & \\
\longrightarrow & H^1(k, \Gamma_{A_s}) & \xrightarrow{\iota} & H^1(k, \text{Aut } (\mathcal{J}_{0s}, A_s^+)) & \xrightarrow{\rho} & H^1(k, \text{Aut } A_s^+) \\
 & \downarrow & & & \\
 & 1 & & &
\end{array}$$

Here the elements of $H^1(k, \text{Aut } (\mathcal{J}_{0s}, A_s^+))$ have a natural interpretation as (isomorphism classes of) k-forms of $(\mathcal{J}_0, A^+)$, i.e., as pairs $(\mathcal{J}, \mathcal{J}')$ of k-algebras that become isomorphic (in the obvious sense) to $(\mathcal{J}_{0s}, A_s^+)$ when extending scalars from k to k_s. A similar interpretation prevails for $H^1(k, \text{Aut } A_s^+)$, and under these interpretations, ρ corresponds to the assignment $(\mathcal{J}, \mathcal{J}') \mapsto \mathcal{J}'$.

5. Lemma *Let $\mu \in k^\times$. Then $\iota \circ \delta(\mu) \in H^1(k, \text{Aut } (\mathcal{J}_{0s}, A_s^+))$ corresponds to the k-form $(\mathcal{J}(A, \mu\mu_0), A^+)$ of $(\mathcal{J}_0, A^+)$.*

Proof. Choose an element $v \in A_s^\times$ satisfying $N_A(v) = \mu$. Then $\alpha : G \to \Gamma_{A_s}, \sigma \mapsto \alpha(\sigma) = v^{-1}{}^\sigma v$, is a 1-cocycle representing $\delta(\mu) \in H^1(k, \Gamma_{A_s})$. Hence $\beta : G \to$ Aut $(\mathcal{J}_{0s}, A_s^+)$ given by

$$\beta(\sigma)\big((a_0, a_1, a_2)\big) = (a_0, a_1{}^\sigma v^{-1} v, v^{-1}{}^\sigma v\, a_2)$$

for $\sigma \in G, a_0, a_1, a_2 \in A$ is a 1-cocycle representing $\iota \circ \delta(\mu) \in H^1(k, \text{Aut } (\mathcal{J}_{0s}, A_s^+))$. Now define $\gamma \in \text{GL}(\mathcal{J}_{0s})$ by $\gamma\big((a_0, a_1, a_2)\big) = (a_0, a_1 v^{-1}, v a_2)$. Then $\beta(\sigma) =$

$\gamma^{-1}\,{}^\sigma\gamma$ $(\sigma \in G)$, and setting $\mathcal{J} = \mathcal{J}(A, \mu\mu_0)$, it is readily checked that $\gamma : (\mathcal{J}_{0s}, A_s^+) \to (\mathcal{J}_s, A_s^+)$ preserves norms and units, hence is an isomorphism. Therefore $(\mathcal{J}, A^+)$ is the k-form of $(\mathcal{J}_0, A^+)$ corresponding to $\iota \circ \delta(\mu)$. □

We are now ready to establish our main result.

6. Theorem (Tits [J, IX Theorem 22], McCrimmon [M2, Theorem 8]) *Let $\mathcal{J}$ be a central simple exceptional Jordan algebra over k containing a subalgebra of the form A^+, where A is a central simple associative k-algebra of degree* 3. *Then there exists a $\mu \in k^\times$ such that the identity transformation of A^+ extends to an isomorphism from $\mathcal{J}$ onto $\mathcal{J}(A, \mu)$.*

Proof. In our previous discussion we set $\mu_0 = 1$. Given a reduced exceptional simple Jordan algebra $\mathcal{K}$, a theorem of Jacobson (see [J, IX Theorem 3], where the restriction to fields of characteristic not two can easily be avoided) asserts that any homomorphism from any simple reduced subalgebra of degree 3 to $\mathcal{K}$ extends to an automorphism. Therefore $(\mathcal{J}, A^+)$ is a k-form of $(\mathcal{J}_0, A^+)$ which, when interpreted in $H^1(k, \mathrm{Aut}\ (\mathcal{J}_0, A^+)$, belongs to kernel of ρ (**4**.), hence to the image of ι. But δ is surjective, so by **5**., some $\mu \in k^\times$ has $(\mathcal{J}(A, \mu), A^+) \cong (\mathcal{J}, A^+)$, as claimed. □

7. Remark. Every exceptional simple Jordan algebra $\mathcal{J}$ admits an important invariant, its trace form, called the "*invariant* mod 2" by Serre ([S2]). In [S2] Serre also attaches a certain decomposable element of $H^3(k, \mathbf{Z}/\mathbf{Z}3)$ to $\mathcal{J}$ and conjectures that this is an invariant as well, called the "*invariant* mod 3". This conjecture has recently been settled affirmatively by Rost ([R]). However, a further conjecture of Serre's (loc. cit.) that exceptional simple Jordan algebras are classified by their invariants mod 2 and 3 is still unsolved.

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