Albert Division Algebras in Characteristic three contain Cyclic Cubic Subfields.

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Abstract. It is shown that a finite-dimensional absolutely simple nonsingular Jordan division algebra of degree 3 over a field containing the third roots of unity admits a cyclic cubic subfield.

The question as to whether every Albert division algebra contains a cyclic cubic subfield, whose significance derives from its connection with twisted compositions in the sense of Springer [2,6], has been raised by Albert [1] and remains unsolved to this day. In the presence of the third roots of unity, an affirmative answer has been given by Petersson-Racine [4] provided the underlying base field has characteristic different from 2 and 3. It is the purpose of the present note to remove this extra hypothesis by establishing the following result.

Theorem. *Let k be a field containing the third roots of unity. Then every Albert division algebra over k contains a cyclic cubic subfield.*

The hypothesis on k being automatic if the characteristic is 3, the statement of the title drops out as an immediate corollary.

We whish to prove the theorem in the more general setting of absolutely simple nonsingular Jordan algebras of degree 3 and begin by summarizing the relevant facts needed for that purpose; the reader ist referred to McCrimmon [3] for details. Let k be an arbitrary field and $\mathcal{J}$ a finite-dimensional unital quadratic Jordan algebra of degree 3 oder k. Then $\mathcal{J}$ arises form a cubic form $N : \mathcal{J} \to k$ (given by its norm) with adjoint $\sharp : \mathcal{J} \to \mathcal{J}$ (given by the numerator of the inversion map) and base point $1 \in \mathcal{J}$ (given by the identity element). We assume throughout that $\mathcal{J}$ is *absolutely simple* (hence remains simple under all base field extensions) and *nonsingular*, i.e., the trace form

$$T = -(D^2 \log N)(1) : \mathcal{J} \times \mathcal{J} \longrightarrow k$$

has this property (which follows from absolute simplicity unless we are in characteristic 2). Every element x in any scalar extension of $\mathcal{J}$ satisfies the generic minimum polynomial

$$\lambda^3 - T(x)\lambda^2 + S(x)\lambda - N(x), \tag{1}$$

where $T(x) = T(x, 1)$ and $S(x) = T(x^\sharp)$. The quadratic form S bilinearizes to

$$\begin{aligned} S(x, y) &= S(x + y) - S(x) - S(y) \\ &= T(x)T(y) - T(x, y). \end{aligned} \tag{2}$$

We write S^0 for the restriction of S to $\mathcal{J}^0 = \ker T$. If char $k \neq 3$, S^0 is nonsingular. If char $k = 3$, rad $S^0 = k1$.

Now suppose $\mathcal{J}$ is reduced and hence, for some composition algebra C over k and some diagonal matrix $g \in \mathrm{GL}_3(k)$, identifies with the Jordan algebra $\mathrm{H}_3(C, g)$ of 3-by-3 g-hermitian matrices with entries in C and scalars down the diagonal. We write e_i $(1 \leq i \leq 3)$ for the absolutely primitive standard idempotents of $\mathcal{J}$ and observe

$$\textstyle\sum e_i = 1, \; e_i^\sharp = 0, \; T(e_i, e_j) = \delta_{ij} \qquad (1 \leq i, j \leq 3). \tag{3}$$

Hence $u_i = e_i - e_3 \in \mathcal{J}^0$ $(i = 1, 2)$, and (2), (3) imply

$$S^0(u_i) = S^0(u_1, u_2) = -1 \qquad (i = 1, 2). \tag{4}$$

Proposition 1. *Assume k has characteristic not 3 and contains the third roots of unity. Then the quadratic form S^0 is isotropic.*

Proof. By Springer's Theorem on quadratic forms [5, Chap. 2, 5.3] (where the restriction to fields of characteristic not two is easily seen to be unnecessary), we may assume that $\mathcal{J}$ is reduced. Then, letting $\zeta \in k$ be a primitive third root of unity, (4) implies

$$S^0(u_1 + \zeta u_2) = -(1 + \zeta + \zeta^2) = 0. \qquad \square$$

Proposition 2. *Assume* char $k = 3$. *Then the quadratic form S^0 represents the element $-1 \in k$.*

Proof. Applying Springer's Theorem again, this time to the quadratic form $q \perp < 1 >$, where q is induced by S^0 on $\mathcal{J}^0/k1$, we may again assume that $\mathcal{J}$ is reduced. But then (4) again immediately yields what we want.

$\square$

We are now ready for the extended version of our original theorem promised earlier.

Theorem 3. *Let k be a field containing the third roots of unity and let $\mathcal{J}$ be a finite-dimensional absolutely simple nonsingular Jordan division algebra of degree 3 over k. Then $\mathcal{J}$ contains a cyclic cubic subfield.*

Proof. For char $k \neq 3$, it suffices to exhibit a Kummer extension of degree 3 in $\mathcal{J}$. By (1) the existence of such an extension is equivalent to S^0 being isotropic, and Proposition 1 completes the proof. For char $k = 3$, the cubic subfield of $\mathcal{J}$ generated by a nonzero element $u \in \mathcal{J}^0$ because of (1) has the discriminant

$$d = -4S^0(u)^3 - 27N(u)^2 = -S^0(u)^3.$$

Hence the existence of cyclic cubic subfields of $\mathcal{J}$ is equivalent to S^0 representing $-1 \in k$, and Proposition 2 completes the proof.

$\square$

Without the hypothesis on k, Theorem 3 is known to be false in dimension 9 (see Petersson-Racine [4, Proposition 5] for a counter example), the case of dimension 27 (Albert algebras) in characteristic not three still being open.

References

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