Structure Theorems for Jordan Algebras of Degree Three over Fields of arbitrary Characteristic

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Dedicated to Kevin McCrimmon

Abstract

Classical results, like the construction of a 3-fold Pfister form attached to any central simple associative algebra of degree 3 with involution of the second kind [HKRT], or the Skolem-Noether theorem for Albert algebras and their 9-dimensional separable subalgebras [PaST], which originally were derived only over fields of characteristic not 2 (or 3), are extended here to base fields of arbitrary characteristic. The methods we use are quite different from the ones originally employed and, in many cases, lead to expanded versions of the aforementioned results that continue to be valid in any characteristic.

0. Introduction

Thanks to their close connection with the Galois cohomology of classical and exceptional groups, Jordan algebras of degree 3 have attracted considerable attention over the last couple of years. The results on the (cohomological) invariants mod 2, based to a large extent on the construction of Haile-Knus-Rost-Tignol [HKRT] attaching a 3-fold Pfister form to any central simple associative algebra of degree 3 with involution of the second kind, are particularly noteworthy in this context, as is the Skolem-Noether theorem of Parimala-Sridharan-Thakur [PaST] for Albert algebras and their 9-dimensional separable subalgebras. Invariably, however, these results, a systematic account of which may be found in [KMRT, §§19, 30, 37-40], are confined to base fields of characteristic not 2; sometimes even characteristic 3 has to be excluded.

In the present paper, an approach to the subject will be developed that yields expanded versions of the aforementioned results over fields of arbitrary characteristic. The methodological framework of our approach is mainly Jordan-theoretic in nature and relies heavily on the Tits process [PR3] for Jordan algebras of degree 3. Another key ingredient is the explicit, characteristic-free description due to Petersson-Racine ([PR6, 1.8, 2.4] and [PR7, 3.8, 3.9]) of the 3-fold Pfister form attached to a central simple associative algebra of degree 3 with involution of the second kind. Finally, the insight, which goes back to Racine [Ra1], that in this generality the role usually played by

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the bilinear trace form is taken over by the *quadratic* trace (cf. 1.4 for the definition) becomes a frequently recurring theme in our investigation.

With special emphasis on those results which seem to be new even when the characteristic is not 2, the content of the paper may be summarized as follows. After covering some background material in Section 1, we proceed to investigate distinguished involutions in the next two sections and show in particular that an involution τ of the second kind on a simple associative algebra B of degree 3 is distinguished if and only if the quadratic trace of $J = H(B,\tau)$, the Jordan algebra of τ-symmetric elements in B, becomes isotropic on the orthogonal complement of any cubic étale subalgebra of J (2.7). We also establish Albert's [A] classical result yielding distinguished cubic subfields of symmetric elements in central simple associative algebras of degree 3 with distinguished involution in all characteristics (3.1). Our proof is different from the one of Haile-Knus [HK] and Villa [KMRT, Ex 19.9] and provides additional information in characteristic 3 (3.9). Section 4 is devoted to Albert algebras and the characteristic-free interpretation of their invariants mod 2 in terms of Pfister forms. The main results are 4.4 and 4.9, characterizing in various ways Albert algebras with vanishing 5- (resp. 3-) invariant mod 2. As an application of 4.4, we obtain examples of Albert division algebras which are isotopic but not isomorphic (4.7). The proof of the Skolem-Noether theorem of [PaST] in arbitrary characteristic will be taken up in Section 5. The paper concludes in 6.5 with comparing the two descriptions of the 3-fold Pfister form attached to an involution given by [HKRT, Proposition 19] (see also [KMRT, (19:25)] and by [PR3, 3.8, 3.9] (see 3.3 below).

1. Background material

1.0 Throughout this paper, we fix a base field k of arbitrary characteristic. All algebras considered in the sequel are assumed to be finite-dimensional and to contain an identity. The set of invertible elements in a structure A will be denoted by $A^\times$, whenever this makes sense. We systematically write $Q(x,y) = Q(x+y) - Q(x) - Q(y)$ for the bilinearization of a quadratic map Q. Basic concepts and facts from the theory of (quadratic) Jordan algebras will be taken for granted, the standard reference being Jacobson [J2]. Given a Jordan algebra J, we write $k[x]$ for the subalgebra of J generated by x. The generic norm of J [JK] will always be viewed as a polynomial function [Rb], acting, mostly under the same notation, on every base change of J in a functorial manner; dito for the other coefficients of the generic minimum polynomial.

The main purpose of this section is to collect a few results scattered in the literature that are indispensable for understanding the subsequent development. Proofs will be omitted most of the time.

1.1 Quadratic forms versus symmetric bilinear forms. Since we do not exclude characteristic 2, we have to distinguish carefully between quadratic forms on the one hand and symmetric bilinear forms on the other. Our basic reference for both are Micali-Revoy [MR] and Scharlau [Scha]. They will be identified only if characteristic 2 has been expressly ruled out. Given a symmetric matrix S of size n, the symmetric bilinear form induced by S on n-dimensional column space k^n will be denoted by $\langle S\rangle$. If $S = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ is diagonal, we write $\langle \gamma_1, \ldots, \gamma_n\rangle = \langle S\rangle$. The hyperbolic plane as a binary quadratic form will be denoted by $\mathbf{h}$. A quadratic form is said to be *nonsingular* if its bilinearization is nondegenerate in the usual sense. For example, the hyperbolic plane $\mathbf{h}$ is nonsingular but, for char $k = 2$ and $\alpha \in k$, the one-dimensional quadratic form $[\alpha]$ is not, even if $\alpha \neq 0$. The Witt classes of nonsingular quadratic forms make up the Witt *group* of k, which is a module over its Witt *ring*, consisting of the Witt classes of nondegenerate symmetric bilinear forms; the corresponding module action is given by the tensor product of a symmetric bilinear form β and a quadratic form q yielding a quadratic form $\beta \,.\, q$ over k. We also recall that Witt's theorem holds for nonsingular quadratic forms, though is doesn't for symmetric bilinear ones even if they are nondegenerate.

1.2 Quadratic étale algebras. Quadratic étale k-algebras are classified by $H^1(k, \mathbb{Z}/2\mathbb{Z})$; in fact, this group will be identified systematically with the group of square classes in $k^\times$ for char $k \neq 2$ and with the cokernel of the Artin-Schreier map $\alpha \mapsto \alpha + \alpha^2$ otherwise. The element of $H^1(k, \mathbb{Z}/2\mathbb{Z})$ corresponding to a quadratic étale k-algebra L will be denoted by $\delta(L/k)$. Conversely, we write $k\{\delta\}$ for the quadratic étale k-algebra corresponding to $\delta \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. Recall that, writing $d_{L/k} \in k^\times/k^{\times 2}$ for the ordinary discriminant of L, we have $d_{L/k} = \delta(L/k)$ for char $k \neq 2$ but $d_{L/k} = 1$ for char $k = 2$.

1.3 Associates of quadratic forms. Let $q : V \to k$ be a nonsingular quadratic form with base point $e \in V$, so $q(e) = 1$. Given $\delta \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ represented by $d \in k^\times$ for char $k \neq 2$ (resp. $d \in k$ for char $k = 2$), $q_\delta(x) = dq(x) + \frac{1-d}{4} q(e,x)^2$ (char $k \neq 2$), $q_\delta(x) = q(x) + dq(e,x)^2$ (char $k = 2$) defines a nonsingular quadratic form $q_\delta : V \to k$ with base point e which up to isometry neither depends on d nor on e [PR1, Proposition 3.1] and is called the δ-*associate* of q. For a discussion of this concept in a much more general setting, see Loos [Lo]. By [PR6, 2.7] we have

$$(q \perp q')_\delta \cong q_\delta \perp \langle d' \rangle . \, q', \tag{1.3.1}$$

where q' is another nonsingular quadratic form and $d' = d$ (char $k \neq 2$), $d' = 1$ (char $k = 2$); also,

$$(q_\delta)_{\delta'} \cong q_{\delta + \delta'} \tag{1.3.2}$$

for $\delta, \delta' \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. Moreover, writing N_L for the norm of an étale k-algebra L, we recall

$$(N_{k\{\delta\}})_{\delta'} \cong N_{k\{\delta + \delta'\}} \tag{1.3.3}$$

from [PR6, 2.9]. Finally,

$$\langle d_{L/k} \rangle . \, N_L \cong \langle -1 \rangle . \, N_L \tag{1.3.4}$$

for any quadratic étale k-algebra L.

1.4 Cubic norm structures. Following McCrimmon [McC], and adopting the terminology of Petersson-Racine [PR3], we define a *cubic norm structure* over k as a quadruple $(V, N, \sharp, 1)$ consisting of a finite-dimensional vector space V over k, a cubic form $N : V \longrightarrow k$ (the *norm*), a quadratic map $\sharp$: $V \longrightarrow V$, $x \longmapsto x^\sharp$, (the *adjoint*) and a distinguished element $1 \in V$ (the *base point*) such that the relations $x^{\sharp\sharp} = N(x)x$ (the *adjoint identity*), $N(1) = 1$, $T(x^\sharp, y) = (DN)(x)y$ (the directional derivative of N at x in the direction y), $1^\sharp = 1$, $1 \times y = T(y)1 - y$ hold under all scalar extensions, where $T := -(D^2 \log N)(1) : V \times V \longrightarrow k$ ist the *associated trace form*, $x \times y = (x+y)^\sharp - x^\sharp - y^\sharp$ is the bilinearization of the adjoint and $T(y) = T(y, 1)$. Then the

U-operator $U_x y = T(x,y)x - x^\sharp \times y$ and the base point 1 give V the structure of quadratic Jordan algebra denoted by $J(V, N, \sharp, 1)$. Among the results obtained in [McC], the following are particularly noteworthy in the present context. Writing $S : J \longrightarrow k$ for the quadratic form given by $S(x) = T(x^\sharp)$, we have the relation

$$x^\sharp = x^2 - T(x)x + S(x)1. \tag{1.4.1}$$

Isomorphisms between Jordan algebras of cubic norm structures with *nondegenerate* associated trace forms are the same as bijective linear maps preserving norms and base points. An element $y \in J$ is invertible if and only if $N(y) \neq 0$, in which case $J^{(y)}$, the y-isotope of J, again arises from a cubic norm structure in an explicit manner. J satisfies not only the cubic equation

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0 \tag{1.4.2}$$

but also, as pointed out in [JK, p. 220], its quartic companion

$$x^4 - T(x)x^3 + S(x)x^2 - N(x)x = 0,$$

the latter being implied by the former only in characteristic not 2. In particular, J has degree at most 3. Conversely, given any Jordan algebra J of degree 3 over k, with generic norm $N = N_J$ and identity element $1 = 1_J$, we define the adjoint $\sharp$ as the numerator of the inversion map to obtain a cubic norm structure $(V, N, \sharp, 1)$ (V being the vector space underlying J) satisfying $J = J(V, N, \sharp, 1)$. We then write $T = T_J$ for the associated trace form, which agrees with the generic trace of J, and call $S = S_J$ the *quadratic trace* of J. For $y \in J^\times$, the quadratic trace of $J^{(y)}$ is given by

$$S_{J^{(y)}}(x) = T_J(y^\sharp, x^\sharp). \tag{1.4.3}$$

Denoting by $J^0 = \ker T$ the space of trace zero elements in J, the relation (cf. [McC, (16)])

$$S(x,y) = T(x)T(y) - T(x,y) \qquad (x, y \in J) \tag{1.4.4}$$

immediately implies the following elementary observation.

1.5 Proposition. *Let J be a Jordan algebra of degree* 3 *over k whose generic trace is nondegenerate. Then S_J^*, defined to be the quadratic trace of J for* char $k \neq 2$ *and its restriction to J^0 for* char $k = 2$, *is a nonsingular quadratic form over k.* □

1.6 Cubic étale algebras. Given a cubic étale k-algebra E, the preceding considerations apply to its associated Jordan algebra, which has degree 3 and will be identified with E. Also, we write $\Delta(E)$ for the discriminant of E, viewed as a quadratic étale k-algebra [KMRT, §18], and $\delta(E/k)$ for the element of $H^1(k, \mathbb{Z}/2\mathbb{Z})$ determined by $\Delta(E)$. The connection between $\delta(E/k)$ and $d_{E/k}$, the ordinary discriminant, is the same as for quadratic étale algebras, cf. 1.2. Sometimes we will use the Scharlau transfer $(T_E)_*$ [Scha, Chap. 2] (cf. [PR7, 3.4, 3.5] for a characteristic-free ad-hoc description) of quadratic (resp. symmetric bilinear) forms over E to quadratic (resp. symmetric bilinear) forms over k. The key fact is Frobenius reciprocity

$$(T_E)_*\big(\beta\,.\,(q \otimes E)\big) \cong (T_E)_*(\beta)\,.\,q \tag{1.6.1}$$

for a symmetric bilinear form β over E and a quadratic form q over k. Dito for étale k-algebras of degree other than 3.

1.7 Reduced algebras. Let J be a Jordan algebra of degree 3 over k. Then J is either a division algebra or it is reduced [Ra1, Theorem 1]. In the latter case, J can be co-ordinatized, so there exist a composition algebra C over k (the possibility of a purely inseparable field extension of characteristic 2 and exponent 1 being included) and a diagonal matrix $g = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3) \in \mathrm{GL}_3(k)$ with the following property: J is isomorphic to $H_3(C, g)$, the Jordan algebra of all 3-by-3 matrices x over C which have diagonal entries in k and are g-hermitian in the sense that $x = g^{-1t}x^*g$, $*$ being the canonical involution of C. We call C, which is unique up to isomorphism, the co-ordinate algebra of J. Writing N_C (resp. T_C) for the norm (resp. trace) of C and

$$u[jl] = \gamma_l u e_{jl} + \gamma_j u^* e_{lj} \qquad (u \in C,\ 1 \le j, l \le 3,\ j \ne l)$$

for the usual hermitian matrix units, the elements of $J = H_3(C, g)$ have the form

$$x = \sum \alpha_i e_i + \sum u_i[jl],\ y = \sum \beta_i e_i + \sum v_i[jl] \qquad (\alpha_i, \beta_i \in k, u_i, v_i \in C), \tag{1.7.1}$$

both multiple sums being extended over all cyclic permutations (ijl) of (123). By [McC, p. 502], norm, adjoint, base point and associated trace form of J are given by the formulae

$$N_J(x) = \alpha_1\alpha_2\alpha_3 - \sum \gamma_j\gamma_l\alpha_i N_C(u_i) + \gamma_1\gamma_2\gamma_3 T_C(u_1u_2u_3), \tag{1.7.2}$$

$$x^\sharp = \sum \big(\alpha_j\alpha_l - \gamma_j\gamma_l N_C(u_i)\big)e_i + \sum \big(\gamma_i(u_ju_l)^* - \alpha_i u_i\big)[jl], \tag{1.7.3}$$

$$1 = \sum e_i, \tag{1.7.4}$$

$$T_J(x, y) = \sum \alpha_i\beta_i + \sum \gamma_j\gamma_l N_C(u_i, v_i). \tag{1.7.5}$$

This implies

$$S_J(x) = \sum \big(\alpha_j\alpha_l - \gamma_j\gamma_l N_C(u_i)\big). \tag{1.7.6}$$

Moreover, the diagonal $E = \sum k e_i \subset J$ is a split cubic étale subalgebra, and, writing $E^\perp$ for its orthogonal complement relative to the generic trace, we conclude $S_J|_{E^\perp} \cong Q_J$, where

$$Q_J := \langle -\gamma_2\gamma_3\rangle . N_C \perp \langle -\gamma_3\gamma_1\rangle . N_C \perp \langle -\gamma_1\gamma_2\rangle . N_C \tag{1.7.7}$$

is an invariant of J [P3, p. 593]. The following standard fact has been observed in [P3, Proposition 1]. (Recall that an algebra is said to be absolutely simple if it stays simple under all base field extensions.)

1.8 Proposition. *Given a reduced absolutely simple Jordan algebra J of degree 3 over k, the following statements are equivalent.*

(i) *J contains nonzero nilpotent elements.*

(ii) *Q_J is isotropic.*

(iii) *J can be co-ordinatized as in 1.7, with $g = \mathrm{diag}(1, -1, 1)$.*

□

1.9 The Tits process. We now recall from [PR3] the most important technical tool of the paper. Let K be a quadratic étale k-algebra, B a separable associative algebra of degree 3 over K (with the obvious meaning if $K \cong k \times k$ is split) and τ a K/k-involution of B. Assume that we are given invertible elements $u \in H(B,\tau)$, the Jordan algebra of τ-symmetric elements in B, and $b \in K$ satisfying $N_B(u) = N_K(b)$. Then we may extend $N_B, \sharp$ (the adjoint of B or B^+, cf. 1.4), 1_B as given on B and $H(B,\tau)$ to the k-vector space $V = H(B,\tau) \times B$ according to the rules

$$N\big((v_0,v_1)\big) = N_B(v_0) + bN_B(v) + \tau(b)\tau\big(N_B(v)\big) - T_B\big(v_0, vu\tau(v)\big), \tag{1.9.1}$$

$$(v_0,v)^\sharp = \big(v_0^\sharp - vu\tau(v),\ \tau(b)\tau(v)^\sharp u^{-1} - v_0 v\big), \tag{1.9.2}$$

$$1 = (1_B, 0) \tag{1.9.3}$$

for $v_0 \in H(B,\tau),\ v \in B$ to obtain a cubic norm structure whose corresponding Jordan algebra will be written as $J = J(K,B,\tau,u,b)$. The associated trace form is given by

$$T(x,y) = T_B(v_0,w_0) + T_B\big(vu,\tau(w)\big) + T_B\big(wu,\tau(v)\big) \tag{1.9.4}$$

for $x = (v_0,v), y = (w_0,w) \in J$. Furthermore, $H(B,\tau)$ identifies as a subalgebra of J through the first factor. We recall from [PR3, 5.2] that J is a division algebra if and only if b is not a generic norm of B. The following useful result has been established in [PR3, 3.7].

1.10 Proposition. *Notations being as in* 1.9, *let* $w \in B^\times$ *and put* $u' = wu\tau(w),\ b' = N_B(w)b$. *Then the assignment* $(v_0,v) \longmapsto (v_0,vw)$ *determines an isomorphism from* $J(K,B,\tau,u',b')$ *onto* $J(K,B,\tau,u,b)$. □

1.11 The second Tits construction. If (B,τ) is a central simple associative algebra of degree 3 over k with involution of the second kind (central simplicity being understood in the category of algebras *with involution*, cf. [KMRT, pp. 20,21]), the Tits process 1.9 applies to $K = \mathrm{Cent}(B)$, the centre of B, and $J(B,\tau,u,b) := J(K,B,\tau,u,b)$ is an Albert algebra, i.e., a k-form of $H_3(\mathrm{Zor}(k)) = H_3(\mathrm{Zor}(k), \mathbf{1}_3)$, where $\mathrm{Zor}(k)$ is the split octonion algebra of Zorn vector matrices [SV, 1.8] and $\mathbf{1}_3$ stands for the 3-by-3 unit matrix. For example, we may choose $u = 1,\ b = 1$, forcing the Albert algebra $J(B,\tau,1,1)$ to be reduced. Following [PR6, 1.7], we write Oct J for the co-ordinate algebra of $J(B,\tau,1,1)$ in the sense of 1.7, which is an octonion algebra called the octonion algebra of $J = H(B,\tau)$. Given any cubic étale subalgebra $E \subset J$, the norm of Oct J can be described by the following formulae (cf. [PR7, 1.11]):

$$N_{\mathrm{Oct}\ J} \cong N_{k\{\delta(K/k)+\delta(E/k)\}} \perp \langle d_{K/k}\rangle . S_J|_{E^\perp}, \tag{1.11.1}$$

$$\langle d_{K/k}\rangle . S_J \cong \langle -1\rangle \perp N_{\mathrm{Oct}\ J} \qquad (\mathrm{char}\ k \neq 2), \tag{1.11.2}$$

$$S_J^0 \cong \big(N_{\mathrm{Oct}\ J}\big)_{\delta(K/k)+1} \qquad (\mathrm{char}\ k \neq 2). \tag{1.11.3}$$

1.12 The first Tits construction. Let A be a separable associative algebra of degree 3 over k and $\alpha \in k^\times$. Then $N_A, \sharp, 1_A$ as given on A extend to the vector space $A \times A \times A$ according to the rules

$$N\big((v_0,v_1,v_2)\big) = N_A(v_0) + \alpha N_A(v_1) + \alpha^{-1}N_A(v_2) - T_A(v_0v_1v_2), \tag{1.12.1}$$

$$(v_0,v_1,v_2)^\sharp = \big(v_0^\sharp - v_1v_2, \alpha^{-1}v_2^\sharp - v_0v_1, \alpha v_1^\sharp - v_2v_0\big), \tag{1.12.2}$$

$$1 = (1_A,0,0) \tag{1.12.3}$$

for $v_0, v_1, v_2 \in A$ to yield a cubic norm structure over k whose associated Jordan algebra will be denoted by $J = J(A, \alpha)$; clearly, A^+ identifies as a subalgebra of J through the first factor. If $K \cong k \times k$ as in 1.9 splits, we obtain $B \cong A \times A^{\mathrm{op}}$ for some separable associative k-algebra A of degree 3 and τ is the exchange involution, allowing us to identify A^+ with $H(B, \tau)$ via the diagonal embedding. Also, $b = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \in k^\times$, and applying 1.10 twice (cf. [PR3, 3.8]) yields an explicit isomorphism $J(K, B, \tau, u, b) \cong J(A, \alpha_1)$ extending the identity of $A^+ = H(B, \tau)$.

1.13 The étale Tits process. Let L, E be étale k-algebras of dimension $2, 3$, respectively, and write σ for the nontrivial k-automorphism of L. Following [PT], we apply 1.9 to $K = L,\ B = E \otimes L$ and $\tau = \sigma$, acting as $\mathbf{1}_E \otimes \sigma$ on B. Hence, given $u \in E = H(E \otimes L, \sigma)$ and $b \in L$ having the same nonzero norms, we may perform the Tits process 1.9 to obtain the algebra $J(E, L, u, b) = J(L, E \otimes L, \sigma, u, b)$. If $L \cong k \times k$ splits, the étale Tits process becomes the étale first Tits construction $J(E, \alpha)$ for some $\alpha \in k^\times$ as in 1.12. Combining [PR2, Theorem 1] with [PT, 1.6] and [PR7, (1.10.2)], we conclude:

1.14 Theorem. *Let L, E be étale k-algebras of dimension $2, 3$, respectively, and (B, τ) a central simple associative algebra of degree 3 with involution of the second kind over k. Given an isomorphic embedding ι from E to $J = H(B, \tau)$, the following statements are equivalent.*

- (i) *There exist invertible elements $u \in E,\ b \in L$ having the same norms such that ι extends to an isomorphism from the étale Tits process $J(E, L, u, b)$ onto J.*
- (ii) *Writing K for the centre of B, we have $\delta(L/k) = \delta(K/k) + \delta(E/k)$ in $H^1(k, \mathbb{Z}/2\mathbb{Z})$.*

In this case, $d_{L/k} = d_{K/k} d_{E/k}$ in $k^\times / k^{\times 2}$. □

1.15 Springer forms. Let J be a Jordan algebra of degree 3 over k and $E \subset J$ a cubic étale subalgebra. Following [PR1, Proposition 2.1], the assignment $(v, x) \longmapsto -v \times x$ gives $E^\perp$ the structure of a left E-module and, for every $x \in E^\perp$, we may decompose $x^\sharp = -q_E(x) + r_E(x),\ q_E(x) \in E, r_E(x) \in E^\perp$ to obtain a quadratic form $q_E : E^\perp \longrightarrow E$, called the Springer form of E in J. We are now in a position to recall [PR4, 3.7].

1.16 Lemma. *Notations being as in 1.15, suppose $y \in E^\perp$ and $q_E(y) \in E$ are both invertible in J. Suppose further that the subalgebra J' of J generated by E and y has dimension 9. Then*

$$L = k[X]/\Big(X^2 - X + N_J(y)^{-2} N_J\big(q_E(y)\big)\Big)$$

is a quadractic étale k-algebra,

$$u = q_E(y) \in E^\times,\ b = N_J(y)(1 - X) \bmod \Big(X^2 - X + N_J(y)^{-2} N_J\big(q_E(y)\big)\Big) \in L^\times$$

have the same norms and $J' \cong J(E, L, u, b)$. □

1.17 Reduced models. Following [PR7, 2.8], every absolutely simple Jordan algebra J of degree 3 over k has a unique reduced model, denoted by J_{red}, which is characterized by the following condition: J_{red} is a reduced absolutely simple Jordan algebra of degree 3 over k as in 1.7 satisfying $J \otimes F \cong J_{\mathrm{red}} \otimes F$ for every field extension F/k which reduces J in the sense that the base change $J \otimes F$ is reduced over F. We then call the co-ordinate algebra of J_{red} the coordinate algebra of J.

2. Involutions and Pfister forms

2.0 Our aim in this section is to extend the construction of [HKRT] (see also [KMRT]) attaching a 3-fold Pfister form to any central simple associative algebra of degree 3 with involution of the second kind to base fields of arbitrary characteristic. With an eye on applications later on, we begin the discussion in a slightly more general setting. Concerning Pfister forms, we follow the notational conventions of [KMRT, p. xxi].

2.1 Jordan algebras of degree 3 and Pfister forms. Let J be an absolutely simple Jordan algebra of degree 3 over k whose generic trace is nondegenerate. (The latter restriction only excludes the symmetric 3-by-3 matrices over k and their isotopes for char $k = 2$.) Writing C for the co-ordinate algebra of J as in 1.17, the reduced model of J has the form $J_{\mathrm{red}} \cong H_3(C, g)$ for some diagonal matrix $g = \mathrm{diag}(-\gamma_1, -\gamma_2, 1) \in \mathrm{GL}_3(k)$ (1.7, 1.17). If $F = k \times k \times k$ stands for the split étale cubic k-algebra, we may combine [PR7, 2.2] with the relations

$$S_E \cong \langle -d_{E/k} \rangle \perp \mathbf{h} \qquad (\text{char } k \neq 2), \tag{2.1.1}$$

$$S_E^0 \cong N_{k\{\delta(E/k)+1\}} \qquad (\text{char } k = 2), \tag{2.1.2}$$

valid for arbitrary cubic étale k-algebras E [PR6, 3.3, 3.2], to conclude that, in the terminology of 1.5,

$$S_J^* \cong S_F^* \perp Q_J, \tag{2.1.3}$$

where

$$Q_J := Q_{J_{\mathrm{red}}} \cong \langle \gamma_1, \gamma_2, -\gamma_1\gamma_2 \rangle . N_C \tag{2.1.4}$$

is as in (1.7.7). Hence

$$N_C \perp \langle -1 \rangle . Q_J \cong \langle\langle \gamma_1, \gamma_2 \rangle\rangle . N_C \tag{2.1.5}$$

is an $(n+2)$-fold Pfister form if C has dimension 2^n.

2.2 The Pfister form of an involution. For the rest of this section, we fix a central simple associative algebra (B, τ) of degree 3 with involution of the second kind over k. We write $K = \mathrm{Cent}(B)$ for the centre of B and $J = H(B, \tau)$ for the Jordan algebra over k of τ-symmetric elements in B. Specializing 2.1 to J, we obtain $C \cong K$, and

$$\beta = \langle -\gamma_1, -\gamma_2, \gamma_1\gamma_2 \rangle \tag{2.2.1}$$

is a nondegenerate symmetric bilinear form of dimension 3 and determinant 1 satisfying

$$S_J^* \cong S_F^* \perp (\langle -1\rangle \beta) \,.\, N_K, \tag{2.2.2}$$

in accordance with [KMRT, (11.22)]. Hence, as in [KMRT, (19.4)],

$$\pi(J) := \pi(\tau) := N_K \perp \beta \,.\, N_K \tag{2.2.3}$$

is a 3-fold Pfister form, called the *Pfister form of* J (or *of* τ). It is clear from the construction that J and J_{red} have isometric Pfister forms. The connection with the octonion algebra of J is the obvious one.

2.3 Proposition. *Notations being as in* 2.2, *the Pfister form and the octonion norm of* J *are isometric.*

Proof. Writing $J_{\mathrm{red}} = H_3(K, g)$ for some $g = \mathrm{diag}(-\gamma_1, -\gamma_2, 1) \in \mathrm{Gl}_3(k)$, we apply (2.2.1), (2.2.3) and obtain $\pi(J) \cong \langle\langle \gamma_1, \gamma_2 \rangle\rangle \,.\, N_K$. On the other hand, by [PR7, 2.7], Oct $J \cong \mathrm{Cay}(K; \gamma_1, \gamma_2)$ as an iterated Cayley-Dickson doubling process, and the assertion follows. □

Our next aim will be to show that the involution τ up to isomorphism is uniquely determined by its Pfister form, thus extending [HKRT, Theorem 15] (or [KMRT, (19.6)]) to base fields of arbitrary characteristic.

2.4 Theorem. *Notations being as in* 2.2, *let* τ' *be another involution of the second kind on* B *and put* $J' = H(B, \tau')$. *Then the following statements are equivalent.*

(i) τ' *and* τ *are conjugate, i.e.,* $\tau = \mathrm{Int}(u) \circ \tau' \circ \mathrm{Int}(u)^{-1}$ *for some* $u \in B^\times$, *where* $\mathrm{Int}(u)$ *stands for the inner automorphism of* B *determined by* u.

(ii) (B, τ') *and* (B, τ) *are isomorphic, i.e., there exists a* k*-automorphism* φ *of* B *satisfying*

$$\varphi \circ \tau' = \tau \circ \varphi.$$

(iii) J' *and* J *are isomorphic.*

(iv) $S_{J'}^*$ *and* S_J^* *are isometric.*

(v) $\pi(J')$ *and* $\pi(J)$ *are isometric.*

(vi) Oct J' *and* Oct J *are isomorphic.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are obvious, whereas (iv) ⇔ (v) ⇔ (vi) follow from Witt cancellation in (2.2.2), (2.2.3) and from 2.3, respectively. We also have (iii) ⇒ (ii) by [J2, Theorem 5 and p. 118]. It therefore suffices to establish the implications (v) ⇒ (i) or (iv) ⇒ (i). For char $k \neq 2$, this is just part of [KMRT, (19.6)], so we may assume char $k = 2$. Moreover, the argument given in [KMRT, p. 305] works in this special case as well providing K or B is split. We are thus allowed to assume that K is a field and B is a division algebra. Then, however, the argument of loc. cit. breaks down, relying as it does on a theorem of Bayer-Fluckiger and Lenstra [BFL, Corollary 1.4] (see also [KMRT, (6.17)]), which has been proved in characteristic not 2 only.

Therefore we are forced to proceed in a different way, and we do so by reducing the case char $k = 2$ to the case char $k \neq 2$ as follows. First we note that (ii) implies (i) since B, being a division algebra of degree 3 over K, does not allow any K-linear anti-automorphisms. Hence it suffices to establish the implication (vi) $\Rightarrow$ (iii). We do so by following Teichmüller [T] to find a local field k_0 of characteristic 0 whose residue class field is our given field k of characteristic 2. In what follows, free use will be made of the noncommutative and nonassociative valuation theory developed in [Schi],[P1],[P2], [P4]. First of all, $(B,\tau),(B,\tau')$ have a unique lift to unramified central associative division algebras $(B_0,\tau_0),(B_0,\tau_0')$, respectively, of degree 3 with involution of the second kind over k_0 [P4, Theorem 1]. Clearly, $K_0 = \mathrm{Cent}(B_0)$ is an unramified quadratic field extension of k_0, with residue class field K, and $J_0 = H(B_0,\tau_0),\ J_0' = H(B_0,\tau_0')$ are unramified Jordan division algebras of degree 3 and dimension 9 over k_0, with residue class algebras J, J', respectively [P4, Proposition 2]. Setting $C = \mathrm{Oct}\ J,\ C' = \mathrm{Oct}\ J',\ C_0 = \mathrm{Oct}\ J_0,\ C_0' = \mathrm{Oct}\ J_0'$, we conclude $C' \cong C$ from (vi) and claim that it suffices to show $C_0' \cong C_0$. Indeed, this implies $J_0' \cong J_0$ since 2.4 is known to hold for k_0, and passing to the residue class algebras gives (iii). In order to prove $C_0' \cong C_0$, we first establish the following intermediate result:

(2.4.1) *If C_0 is split, so is C. If C_0 is a division algebra, then C is the residue class algebra of C_0, hence a division algebra as well.*

To do so, we begin by recalling that C_0 is the co-ordinate algebra of the reduced Albert algebra $\mathcal{J}_0 = J(B_0,\tau_0,1,1)$ (1.11). Furthermore, we write $\mathfrak{o}_0$ for the valuation ring of k_0, $\mathfrak{p}_0$ for the valuation ideal of $\mathfrak{o}_0$, $\mathfrak{O}_0$ for the valuation ring of K_0 and M_0 for the valuation ring of B_0. Then τ_0 restricts to an $\mathfrak{O}_0/\mathfrak{o}_0$-involution of M_0, also written as τ_0, such that $H(M_0,\tau_0)$ is the valuation ring of J_0. Furthermore, extending the terminology of the Tits process (1.9) to the arithmetic setting in the obvious way,

$$\mathcal{M}_0 := J(M_0,\tau_0,1,1) = H(M_0,\tau_0) \times M_0 \subset \mathcal{J}_0$$

turns out to be an $\mathfrak{o}_0$-order of $\mathcal{J}_0$ which, thanks to a theorem of Brühne [Br, 3.9.10], is selfdual in the sense that it agrees with its dual lattice relative to the trace form. Reduction mod $\mathfrak{p}_0$ gives $\mathcal{M}_0 \otimes k \cong J(B,\tau,1,1)$, so

(2.4.2) *C is the co-ordinate algebra of $\mathcal{M}_0 \otimes k$.*

We now distinguish the following cases.

Case 1. C_0 is split.
Then $C_0 \cong \mathrm{Zor}(k_0)$ is the algebra of Zorn vector matrices over k_0, and since $\mathcal{M}_0$, being selfdual, is distinguished as an order in $\mathcal{J}_0$ (cf. Knebusch [Kn, §8], Racine [Ra1, IV §4 and Lemma 2]), we conclude $\mathcal{M}_0 \cong H_3(\mathrm{Zor}(\mathfrak{o}_0))$ [Ra2, IV Proposition 5]. Reducing mod $\mathfrak{p}_0$ and comparing with (2.4.2) implies $C \cong \mathrm{Zor}(k)$, hence the first part of (2.4.1).

Case 2. C_0 is a division algebra.
Writing R_0 for the valuation ring of C_0, it follows from [Br, 3.4.7] that $\mathcal{M}_0$ is isomorphic to $H_3(R_0, g_0)$ for some diagonal matrix $g_0 \in \mathrm{GL}_3(\mathfrak{o}_0)$. Again reducing mod $\mathfrak{p}_0$ and observing (2.4.2), we obtain the second part of (2.4.1).

Noting that, by symmetry, (2.4.1) holds für C', C_0' as well, we are now ready to prove that C_0' and C_0 are isomorphic. Indeed, if $C' \cong C$ are both split, so are C_0', C_0 by (2.4.1), and hence they are isomorphic. On the other hand, if $C' \cong C$ are both division algebras, (2.4.1) implies that C_0', C_0 are unramified octonion division algebras over k_0 whose residue class algebras are isomorphic. But this implies $C_0' \cong C_0$ by [P2, Theorem 1], as desired. $\square$

2.5 Remark. a) R.S. Garibaldi has proposed an approach to the implication (v) $\Rightarrow$ (i) of 2.4 that is different from the one adopted here and relies on the Rost invariant of algebraic groups (cf. Garibaldi-Merkurjev-Serre [GaMS] or Gille [Gi] for details). We merely sketch the main ingredients of his approach. Following [Ga], the desired implication is equivalent to the Rost invariant having trivial kernel for groups of typ A_2. To prove the latter, [KMRT, (19.6)] does the job for char $k \neq 2$. Otherwise we choose k_0 as above and use Bruhat-Tits theory to lift a group of type A_2 over k to a group of type A_2 over k_0. Since char $k_0 = 0$, the Rost invariant of the latter has trivial kernel. This property being preserved under passage from k_0 to k [Gi, Théorème 2], the assertion follows.
b) A substantial part of the preceding result may be phrased in purely Jordan-theoretical terms as follows. Let J, J' be absolutely simple Jordan algebras of degree 3 and dimension 9 over k and suppose they are isotopic. Then statements (iii) - (vi) of 2.4 are equivalent.

$\square$

2.6 Distinguished involutions. The involution τ of B is said to be *distinguished* if $\pi(\tau)$ is hyperbolic or, what amounts to the same, the octonion algebra of $J = H(B, \tau)$ is split (2.3). By 2.4, distinguished involutions are unique up to conjugation, and over a finite field every involution of the second kind on an algebra of degree 3 is distinguished. Before we can establish the existence of distinguished involutions in general, we require a preparation and define the *index* of a (possibly singular) quadratic form q over k, denoted by $\mathrm{ind}(q)$, as the maximal dimension of totally isotropic subspaces of q. This is clearly the ordinary Witt index if q is nonsingular or k has characteristic not 2.

2.7 Theorem. *Notations being as in* 2.2, *let $E \subset J$ be a cubic étale subalgebra. Then the following statements are equivalent.*

(i) *τ is distinguished.*

(ii) *J_{red} contains nonzero nilpotent elements.*

(iii) $S_J \cong [-d_{K/k}] \perp 4\mathbf{h}$.

(iv) $\mathrm{ind}(S_J) \geq 4$.

(v) *$S_J|_{E^\perp}$ is isotropic.*

(vi) $\mathrm{ind}(S_J^0) \geq 3$.

(vii) $\mathrm{ind}(S_J^0) \geq 2$.

Proof. The equivalence of (i) - (iv) has been established in [PR7, 2.11]. Furthermore, counting dimensions of totally isotropic subspaces we see that (iv) implies (v) and (vi). Since the implication (vi) $\Rightarrow$ (vii) is obvious, it therefore remains to prove that both (v) and (vii) imply (i).
(v) $\Longrightarrow$ (i). The property of S_J to be isotropic on $E^\perp$ by (1.11.1) carries over to $N = N_{\mathrm{Oct}\, J}$.
(vii) $\Longrightarrow$ (i). We first assume char $k \neq 2$. Then (vii) combines with (1.11.1) to show that $\langle -1 \rangle \perp N$ has index ≥ 2, forcing N to be isotropic. On the other hand, assuming char $k = 2$ and setting $\delta = \delta(K/k) + 1$ in $H^1(k, \mathbb{Z}/2\mathbb{Z})$, (vii) combines with (1.11.3) to show that the δ-associate of N has index at least 2. But since N and N_δ agree on a linear hyperplane (1.3), N itself must be isotropic, and we are done again. $\square$

2.8 Remark. For char $k \neq 2, 3$, the equivalence of (i), (vi), (vii) may be found in [KMRT, (19.10)]; the characterization of distinguished involutions by means of (v) seems to be new. $\square$

2.9 Lemma. *Let E be a cubic étale k-algebra and $u \in E^\times$. Then the quadratic form*

$$q_u : E \longrightarrow k,\ x \longmapsto q_u(x) := T_E(u, x^\sharp)$$

is isometric to $\langle N_E(u)\rangle . S_E$. In particular, E contains an invertible element v satisfying $q_u(v) = 0$ unless $k = \mathbb{F}_2$ and $E \cong k \times k \times k$ splits.

Proof. The assignment $x \longmapsto xu$ gives an isometry of the desired kind. Therefore, since the quadratic trace of E satisfies the remaining assertion of the lemma [PR2, Lemma 3], so does q_u. $\square$

2.10 Theorem. *Notations being as in 2.2, let $E \subset J$ be a cubic étale subalgebra. Then there exists an invertible element $v \in E$ such that $\mathrm{Int}(v) \circ \tau$ is a distinguished involution of B. In particular, B admits distinguished involutions fixing E.*

Proof. By 1.14 there exist a quadratic étale k-algebra L as well as invertible elements $u \in E,\ b \in L$ satisfying $N_E(u) = N_L(b)$ such that $J \cong J(E, L, u, b)$. The case of a finite field being obvious (put $v = 1$), we may assume that k is infinite, whence 2.9 yields an invertible element $v \in E$ satisfying $T_E(uv^\sharp) = 0$. Since $H(B, \mathrm{Int}(v) \circ \tau)$ is isomorphic to $J^{(v)}$, the v-isotope of J, and $E^\perp$ agrees with the orthogonal complement of $E^{(v)}$ in $J^{(v)}$, it suffices to show that $S_{J^{(v)}}$ is isotropic on $E^\perp$ (2.7). Setting $y = (0, 1) \in E^\perp$, this follows from

$$\begin{aligned} S_{J^{(v)}}(y) &= T_J\big((v^\sharp, 0),\ (0, 1)^\sharp\big) && \text{(by (1.4.3))} \\ &= T_J\big((v^\sharp, 0), (-u, \tau(b)u^{-1})\big) && \text{(by (1.9.2))} \\ &= -T_E(uv^\sharp) && \text{(by (1.9.4))} \\ &= 0. && \square \end{aligned}$$

2.11 Corollary. (cf. [HKRT, Proposition 17] or [KMRT, (19:30)]) *Let τ be a distinguished involution of B and $E \subset B$ a cubic étale k-subalgebra. Then $H(B, \tau)$ contains a subalgebra isomorphic to E.*

Proof. By 2.4, it suffices to establish the existence of a distinguished involution τ_1 of B such that $H(B, \tau_1)$ contains E. In order to do so, we note that either $k = \mathbb{F}_2$ and $E \cong k \times k \times k$ splits or $E = k[x]$ for some $x \in E$. While in the former case the assertion is obvious, we may invoke [KMRT, (4.18)] in the latter to find a K/k-involution τ' of B satisfying $E \subset H(B, \tau')$. Now 2.10 applies and proves what we want. $\square$

3. Distinguished Involutions and étale first Tits constructions

3.0 As in the preceding section, we let (B, τ) be a central simple associative algebra of degree 3 with involution of the second kind over k and put $K = \mathrm{Cent}(B),\ J = H(B, \tau)$. We will be concerned with the interplay between distinguished involutions and étale first Tits constructions. More specifically, we wish to extend the description of this interplay given in [HKRT, Theorem 16] and [KMRT, (19.14), (19.15), Ex. 19.9] to base fields of arbitrary characteristic as follows.

3.1 Theorem. *Notations being as in* 3.0, *the following statements are equivalent.*

(i) τ *is distinguished.*

(ii) *There exists a cubic étale subalgebra* $E \subset J$ *satisfying* $\delta(E/k) = \delta(K/k)$.

(iii) *There exist a cubic étale subalgebra* $E \subset J$ *and* $\alpha \in k^\times$ *such that* $J \cong J(E, \alpha)$ *is an étale first Tits construction.*

This theorem answers a question raised by Petersson-Racine [PR7, 2.12] in all characteristics. While the implication (iii) $\Rightarrow$ (i) has already been derived in [PR7, 2.11], (ii) $\Rightarrow$ (iii) follows immediately from 1.14 and 1.12. It therefore remains to establish the implication (i) $\Rightarrow$ (ii), which is a difficult result originally due to Albert [A] for char $k \neq 2, 3$. Another approach working in characteristic 2 as well (but still excluding characteristic 3) more recently has been devised by Haile-Knus [HK]; combined with [KMRT, Ex. 19.9] it yields 3.1 in all characteristics. On the other hand, the approach adopted here yields a few additional results of independent interest.

3.2 Lemma. *Assume that* τ *is distinguished and* J *is reduced. Then, given* $\delta \in H^1(k, \mathbb{Z}/2\mathbb{Z})$, *there exists a cubic étale subalgebra* $E \subset J$ *satisfying* $\delta(E/k) = \delta$.

Proof. By 2.11, it suffices to find $g = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3) \in \mathrm{GL}_3(k)$ such that $J' = H_3(K, g)$ contains a cubic étale subalgebra of the desired kind. We may clearly assume $\delta \neq 0$. Given any g as above, Faulkner's Lemma [F, Lemma 1.5] immediately adapts to the present set-up and shows that the Peirce-0-component of J' relative to the diagonal idempotent e_1 is given by

$$J'_0(e_1) = J(S'_0, e_2 + e_3),$$

where S'_0 is the quadratic trace of J' restricted to $J'_0(e_1)$ and the right-hand side refers to the Jordan algebra of a quadratic form with base point. Using (1.7.6), it now follows easily that the minimum polynomial of

$$x = e_2 + 1[23] \in J'_0(e_1)$$

in $J'_0(e_1)$ is $X^2 - X - \gamma_2\gamma_3$. Writing M for the subalgebra of $J'_0(e_1)$ generated by x, $E := ke_1 \oplus M \subset J'$ is a 3-dimensional subalgebra satisfying $\delta(E/k) = 1 + 4\gamma_2\gamma_3$ modulo invertible squares for char $k \neq 2$ and $\delta(E/k) = \gamma_2\gamma_3$ modulo Artin-Schreier elements for char $k = 2$. In any event, choosing γ_2, γ_3 appropriately, we obtain $\delta(E/k) = \delta$, and the proof is complete. $\square$

3.3 Proposition. *Notations being as in* 3.0, *we realize* $J \cong J(E, L, u, b)$ *as an étale Tits process, where* L, E *are étale* k*-algebras of dimension* $2, 3$, *respectively, and* $u \in E$, $b \in L$ *are invertible elements satisfying* $N_E(u) = N_L(b)$. *Then the Pfister form of* J *satisfies*

$$\begin{aligned}\pi(J) &\cong N_L \perp \langle d_{E/k}\rangle (T_E)_*(\langle u\rangle) \,.\, N_L \\ &\cong N_L \perp \langle d_{E/k}\rangle \,.\, (T_E)_*\big(\langle u\rangle \,.\, (N_L \otimes E)\big).\end{aligned}$$

Proof. This follows at once from Frobenius reciprocity (1.6.1) combined with [PR7, 3.8, 3.9] and (1.3.4). □

3.4 Lemma. *Hypotheses being as in* 3.3, *let us assume that J is a division algebra. Given a nonzero element $x \in E \otimes L$, the minimum polynomial of $y := (0, x) \in J$ is $\mu_y = X^3 + pX + q$, where*

$$p = -T_E\big(uN_L(x)\big), \; q = -T_L\big(bN_E(x)\big).$$

Furthermore, $F = k[y] \subset J$ is a cubic subfield satisfying

$$d_{F/k} = 4T_E\big(uN_L(x)\big)^3 - 27T_L\big(bN_E(x)\big)^2$$

modulo nonzero squares and, if $\operatorname{char} k = 2$,

$$\delta(F/k) = \frac{T_E\big(uN_L(x)\big)^3 + T_L\big(bN_E(x)\big)^2}{T_L\big(bN_E(x)\big)^2}$$

modulo Artin-Schreier elements.

Proof. That μ_y has the form as indicated follows immediately from (1.4.2) combined with 1.9. The formula for the discriminant being standard, we are left with the final formula in characteristic 2, which follows from [KMRT, p. 301] or [PR6, 3.6]. □

Dealing with distinguished involutions has the technical advantage of allowing some control over the discriminant of cubic étale subalgebras. This is mainly due to 3.4 and the following fact.

3.5 Lemma. *Hypotheses being as in* 3.3, *let us assume that τ is distinguished. Then*

$$(T_E)_*\big(\langle u\rangle \,.\, (N_L \otimes E)\big) \cong (T_E)_*(\langle u\rangle) \,.\, N_L \cong \; \langle d_{K/k}\rangle \,.\, N_L + 2\mathbf{h}.$$

Furthermore, if E and L are fields, every $d \in K$ yields invertible elements $u_1 \in E, \; b_1 \in L$ satisfying $N_E(u_1) = N_L(b_1), \; J \cong J(E, L, u_1, b_1)$ and $T_E(u_1) = d$.

Proof. Since

$$\pi(J) \cong N_L \perp \langle d_{E/k}\rangle (T_E)_*(\langle u\rangle) \,.\, N_L \qquad \text{(by 3.3)}$$

is hyperbolic, we obtain

$$\langle d_{E/k}\rangle (T_E)_*(\langle u\rangle) \,.\, N_L = \; \langle -1\rangle \,.\, N_L$$

in the Witt group of k, hence

$$\begin{aligned} (T_E)_*(\langle u\rangle) \,.\, N_L &= \; \langle -d_{E/k}\rangle \,.\, N_L \\ &= \; \langle d_{K/k}\rangle \; \langle -d_{L/k}\rangle \,.\, N_L && \text{(by 1.14)} \\ &= \; \langle d_{K/k}\rangle \,.\, N_L && \text{(by (1.3.4)).} \end{aligned}$$

Comparing dimensions, yields the desired formula. In particular, the quadratic form

$$(T_E)_*\big(\langle u\rangle \,.\, (N_L \otimes E)\big)$$

being isotropic, must be universal, so some $w \in (E \otimes L)^\times$ satisfies $T_E\big(uN_L(w)\big) = d$. Setting $u_1 := uN_L(w), \; b_1 := bN_E(w)$ and invoking 1.10 completes the proof. □

3.6 Proposition. *Notations being as in* 3.0, *assume that* τ *is distinguished and* k *has characteristic not* 3. *Then there exists a cubic étale subalgebra* $F \subset J$ *satisfying* $\delta(F/k) = \delta(K_0/k)$ *where*

$$K_0 = k[X]/(X^2 - X + 1).$$

Proof. By 3.2 and 1.14, we may assume that $J \cong J(E, L, u, b)$ is a division algebra arising from the étale Tits process as in 3.3. Then 3.5 yields a nonzero element $x \in E \otimes L$ satisfying $T_E\big(uN_L(x)\big) = 0$. Putting $y = (0, x) \in J$ and $F = k[y] \subset J$, 3.4 implies $d_{F/k} = -3 = d_{K_0/k}$ in $k^\times/k^{\times 2}$ and $\delta(F/k) = 1 = \delta(K_0/k)$ in $k/\wp(k)$ for char $k = 2$. □

3.7 Proof of 3.1, (i) $\Longrightarrow$ (ii) for char $k \neq 3$. By 3.2 we may assume that J is a division algebra. By 3.6 combined with 1.14, we may further assume that $J \cong J(E, L, u, b)$ arises from the étale Tits process as in 3.3 where E satisfies $\delta(E/k) = \delta(K_0/k)$. After these reductions, we apply [PR7, 4.1] to conclude that some isotope of J is a unital Tits process. More precisely, there exists a K/k-involution τ' of B satisfying $J' := H(B, \tau') \cong J(E, L, 1, b')$ for some $b' \in L$ having norm 1, but not belonging to $k1$ since, otherwise, J' would be reduced. Applying 3.4 to $x = 1 \in E \otimes L$, we now obtain a separable cubic subfield $F \subset J'$ which satisfies

$$d_{F/k} = -3\big(T_L(b')^2 - 4N_L(b')\big) = d_{K_0/k} d_{L/k}$$

in $k^\times/k^{\times 2}$. Furthermore, for char $k = 2$, we obtain

$$\delta(F/k) = 1 + \frac{N_L(b')}{T_L(b')^2} = \delta(K_0/k) + \delta(L/k)$$

in $k/\wp(k)$. Hence in all characteristics $\neq 3$ we have $\delta(F/k) = \delta(E/k) + \delta(L/k) = \delta(K/k)$ (by 1.14), so J' is an étale first Tits construction arising from F. But this implies that τ' is distinguished ((iii) $\Rightarrow$ (i) of 3.1), forcing J and J' to be isomorphic. Hence, as J' satisfies 3.1 (ii), so does J. □

It remains to prove the implication (i) $\Rightarrow$ (ii) of 3.1 for char $k = 3$. Actually, the following generalization can be established in this special case.

3.8 Theorem. *Notations being as in* 3.0, *let us assume that* τ *is distinguished and* k *has characteristic* 3. *Then one of the following holds.*

a) *$K \cong k \times k$ splits, and every separable cubic subfield of J is cyclic.*

b) *Every $d \in k$ allows a cubic subalgebra of J having discriminant $d \bmod k^{\times 2}$.*

Proof. If J is reduced, it contains nilpotent elements other than zero (2.7), hence b) holds (3.2). We may therefore assume that J is a division algebra. Since J contains cyclic cubic subfields [P6, Theorem 3], it suffices to establish the following claim.
Either b) *holds or every cubic subfield of J has discriminant $d_{K/k} \bmod k^{\times 2}$.*
To prove this, suppose $E \subset J$ is a separable cubic subfield satisfying $d_{E/k} \neq d_{K/k}$ in $k^\times/k^{\times 2}$. Then $J \cong J(E, L, u, b)$ arises from the étale Tits process as in 3.3, and L is a field (1.14). Moreover, given $d \in k$, we may assume $T_E(u) = -dd_{L/k}$ (3.5). Choosing $\theta \in L$ satisfying $L = k[\theta]$, $\theta^2 = d_{L/k}$, we may apply 3.4 to $x = 1 \otimes \theta \in E \otimes L$ to obtain a cubic subfield $F \subset J$ such that $d_{F/k} = T_E\big(uN_L(x)\big) = d$ in $k^\times/k^{\times 2}$. Hence b) holds, and the proof is complete. □

3.9 Remark. a) If J as in 3.8 is a division algebra, conditions a) and b) hold simultaneously if and only if k is quadratically closed.
b) As observed in [HK], Wedderburn's theorem on the cyclicity of central associative division algebras D of degree 3 is a special case of 3.1: Put $B = D \times D^{\mathrm{op}}$ and let τ be the exchange involution. $\square$

4. Albert Algebras and Pfister forms.

4.0 In this section, we will be concerned with the invariants mod 2 of Albert algebras in arbitrary characteristic. The cohomological interpretation of these invariants (cf. Serre [Se] and [KMRT]), which has to be modified in characteristic 2 along the general lines indicated in [GaMS], will not be discussed here any further. Instead, we rely exclusively on their description by means of Pfister forms. Throughout this section, J will be an arbitrary Albert algebra over k.

4.1 The invariants mod 2 as Pfister forms. Specializing 2.1 to J as in 4.0, the co-ordinate algebra C of J is an octonion algebra over k, and the norm of C, i.e.,

$$\pi_3(J) = N_C, \tag{4.1.1}$$

is a 3-fold Pfister form, called the 3-*invariant* mod 2 *of* J. If $J \cong J(B, \tau, u, b)$ arises from a central simple associative algebra (B, τ) of degree 3 with involution of the second kind by means of the Tits process as in 1.9, where $u \in H(B, \tau), b \in \mathrm{Cent}(B)$ are invertible elements having the same norms, $\pi_3(J)$ becomes isometric to the Pfister form of $\tau^{(u)} = \mathrm{Int}(u) \circ \tau$ ([PR6, 1.8] combined with 2.3, or [KMRT, (40.2)]). On the other hand, returning to 2.1, in particular (2.1.5),

$$\pi_5(J) = \langle\langle \gamma_1, \gamma_2 \rangle\rangle \,.\, N_C \cong N_C \perp \langle -1 \rangle \,.\, Q_J \tag{4.1.2}$$

is a 5-fold Pfister form, called the 5-*invariant* mod 2 *of* J. Clearly, the invariants mod 2 of J and J_{red} are the same. Also, by Racine's characteristic-free version [Ra1, Theorem 3] of Springer's criterion [Sp], combined with the connection between S_J and Q_J (1.7), two *reduced* Albert algebras are isomorphic if and only if they have the same invariants mod 2. For char $k \neq 2$, the Arason invariants $f_i(J)$ of $\pi_i(J)$ $(i = 3, 5)$ yield the *cohomological* invariants mod 2 of J: $f_i(J) \in H^i(k, \mathbb{Z}/2\mathbb{Z})$.

Our principal aim in this section is to give various characterizations of those Albert algebras (some of) whose invariants mod 2 are hyperbolic. Since the 5-invariant is a multiple of the 3-invariant, by (4.1.1), (4.1.2), its hyperbolicity is the weaker condition of the two and will therefore be discussed first. We begin with two simple technicalities.

4.2 Lemma. *Let J' be an absolutely simple Jordan algebra of degree* 3 *over k whose generic trace is nondegenerate and $E \subset J'$ a cubic étale subalgebra. Then*

$$\mathbf{h} \perp Q_{J'} \cong N_{\Delta(E)} \perp (S_{J'}|_{E^\perp}).$$

Proof. Since both sides have the same dimension, it suffices to carry out the following computations in the Witt group of k. We first assume char $k \neq 2$. Then (2.1.1), (2.1.3) imply

$$\begin{aligned} \langle -1 \rangle + Q_{J'} = \; & S_{J'} = S_E + (S_{J'}|_{E^\perp}) \\ = \; & \langle -d_{E/k} \rangle + \; (S_{J'}|_{E^\perp}), \end{aligned}$$

and, adding $\langle 1\rangle$ to both sides, the assertion follows. We are left with the case char $k = 2$. Then (2.1.2), (2.1.3) yield

$$\begin{aligned} N_{k\{1\}} + Q_{J'} &= S^0_{J'} = S^0_E + (S_{J'}|_{E^\perp}) \\ &= N_{k\{\delta(E/k)+1\}} \perp (S_{J'}|_{E^\perp}), \end{aligned}$$

and, passing to the 1-associates of both sides, (1.3.1), (1.3.2) lead to the desired conclusion. $\square$

The second technicality is an exercise in characteristic 2.

4.3 Lemma. *Let R be a commutative associative k-algebra of degree 2 whose generic trace is identically zero, $g \in \mathrm{GL}_3(k)$ a diagonal matrix and $E \subset J' = H_3(R, g)$ a cubic étale subalgebra. For an element $y \in J'$, orthogonal to E relative to the generic trace, to be nilpotent it is necessary and sufficient that $S_{J'}(y) = 0$.*

Proof. The condition is clearly necessary. Conversely, suppose $S_{J'}(y) = 0$. After a suitable base field extension we may assume that $E \cong k\times k\times k$ is split. The relations $N_K(u, v) = T_K(u, v) = 0$ for all $u, v \in R$ imply that R is a local k-algebra, its residue field being purely inseparable of exponent 1 over k. Hence J' is simple modulo its radical, and every complete orthogonal system of absolutely primitive idempotents in J' is connected. Since R, thanks to the Isotopy Theorem of [P5], is an invariant of J', reco-ordinatizing if necessary allows us to assume that E sits diagonally in J'. Extending the notational conventions of 1.7 to the present more general set-up, and representing y as in (1.7.1), we conclude $\beta_i = 0$ for all i, $T_{J'}(y) = 0$ (since $y \in E^\perp$), $S_{J'}(y) = 0$ (by hypothesis) and $N_{J'}(y) = \gamma_1\gamma_2\gamma_3 T_K(v_1v_2v_3)$ (by (1.7.2)) $= 0$. Hence (1.4.2) shows that y is nilpotent. $\square$

4.4 Theorem. *Notations being as in* 4.0, *let $E \subset J$ be a cubic étale subalgebra. Then the following statements are equivalent.*

(i) *$\pi_5(J)$ is hyperbolic.*

(ii) $\mathrm{ind}(Q_J) \geq 8$.

(iii) *Q_J is isotropic.*

(iv) $\mathrm{ind}(S_J) \geq 9$ *for* char $k \neq 2$, $\mathrm{ind}(S^0_J) \geq 8$ *for* char $k = 2$.

(v) $\mathrm{ind}(S^*_J) \geq 3$.

(vi) *J_{red} contains nonzero nilpotent elements.*

(vii) $\mathrm{ind}(S_J|_{E^\perp}) \geq 7$.

(viii) *$S_J|_{E^\perp}$ is isotropic.*

(ix) *There exist a central simple associative algebra (B, τ) of degree 3 with distinguished involution of the second kind and invertible elements $u \in B$, $b \in \mathrm{Cent}(B)$ having the same norms such that $J \cong J(B, \tau, u, b)$.*

Proof. (i) $\Longrightarrow$ (ii). Since $\langle -1 \rangle \,.\, Q_J$ is a subform of $\pi_5(J)$ ((4.1.2)), this follows by counting dimensions of totally isotropic subspaces.
The implications (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v), (vii) $\Rightarrow$ (viii) are clear.
(iii) $\Longrightarrow$ (i). (4.1.2) shows that the Pfister form $\pi_5(J)$ is isotropic, hence hyperbolic.
(ii) $\Longrightarrow$ (iv). This follows immediately from (2.1.3), (2.1.1).
(v) $\Longrightarrow$ (iii). For char $k = 2$, we conclude from (2.1.3) that Q_J is isotropic. For char $k \neq 2$, (2.1.1), (2.1.3) give $\langle 1 \rangle \perp S_J \cong 2\mathbf{h} \perp Q_J$, and we arrive at the same conclusion.
(iii) $\Longleftrightarrow$ (vi). Since $Q_J \cong Q_{J_{\mathrm{red}}}$, this is simply a restatement of 1.8.
(ii) $\Longrightarrow$ (vii). Assuming (ii) and combining 4.2 with a dimension count of totally isotropic subspaces yields (vii).
(viii) $\Longrightarrow$ (ix). This requires a bit more effort. It obviously suffices to show the existence of some (B, τ) as in (ix) such that $H(B, \tau)$ is isomorphic to a subalgebra of J. We begin by choosing a nonzero element $y \in E^\perp$ satisfying $S_J(y) = 0$ and denote by J' the unital subalgebra of J generated by E and y. Then $\dim_k J' \leq 9$. We now consider the following cases.

Case 1. J' is a division algebra.
Then J' is absolutely simple of degree 3 and dimension 9. Since S_J becomes isotropic on $E^\perp \cap J'$, we conclude from 2.7 that J' has the form $H(B, \tau)$ for some central simple associative algebra (B, τ) of degree 3 with distinguished involution of the second kind.

Case 2. J' is not a division algebra.
Then J is reduced, and our first aim will be to show that it contains nonzero nilpotent elements. Assume the contrary. Then the base field is infinite and J' is semisimple of degree 3. If J' were not simple, it would have the form $J' = kc \oplus J''$ as a direct sum of ideals for some nonzero idempotent $c \in J'$ and some Jordan algebra J'' corresponding to a nondegenerate quadratic form with base point [Ra1, Theorem 1]. But since k is infinite and hence J' is generated by two elements, so is J''. This implies $\dim_k J'' \leq 3$, and we conclude $J' = E \oplus ky$. Passing now to the separable closure k_s of k, we may co-ordinatize $J \otimes k_s \cong H_3(C_s)$, C_s being the split octonions over k_s, in such a way that $E \otimes k_s$ corresponds to the diagonal matrices. Then $y = y \otimes 1 \in J' \otimes k_s$ belongs to the sum of off-diagonal Peirce spaces relative to the complete orthogonal system of diagonal idempotents. All Peirce components of y belonging to $J' \otimes k_s$ as well, we may in fact assume $y = v_1[23]$ for some $v_1 \in C_s$ since $\dim J' = 4$. Writing N_s for the norm of C_s, we now conclude $N_s(v_1) = -S_J(y) = 0$ from (1.7.6), and (1.7.3) implies $y^\sharp = 0$, hence $y^2 = y^\sharp + T_J(y)y - S_J(y)1$ (by (1.4.1)) $= 0$, contradicting the absence of nilpotent elements in J. This contradiction shows that J' is simple of dimension at most 9. Hence there exists a two-dimensional composition algebra K over k and a diagonal matrix $g \in \mathrm{GL}_3(k)$ satisfying $J' \subset J_1 := H_3(K, g) \subset J$. By construction, S_J becomes isotropic on $E^\perp \cap J_1$. Hence it follows from 2.7 if K is étale, and from 4.3 otherwise, that $J_1 \cong J_{1\mathrm{red}}$ contains nonzero nilpotent elements, again a contradiction. We have thus shown that J itself contains nonzero nilpotent elements, allowing us to co-ordinatize it as $J \cong H_3(C, g)$, C being the coordinate algebra of J and $g = \mathrm{diag}(1, -1, 1)$ (1.8). Let K be any quadratic étale subalgebra of C. Then $J_1 \cong H_3(K, g)$ contains nonzero nilpotent elements and hence corresponds to a central simple associative algebra of degree 3 over k with distinguished involution of the second kind.
(ix) $\Longrightarrow$ (vi). We put $J' := H(B, \tau)$. Since τ is distinguished, J'_{red} contains nonzero nilpotent elements (2.7) and hence may be co-ordinatized as $J'_{\mathrm{red}} \cong H_3(K, g)$ for some quadratic étale k-algebra K and $g = \mathrm{diag}(1, -1, 1)$. Writing C for the co-ordinate algebra of J, we conclude that $J_{\mathrm{red}} \cong H_3(C, g)$ [PR7, 2.5] contains nonzero nilpotent elements as well. $\square$

4.5 Remark. Assume char $k \neq 2$. Then the equivalence of (i), (ix) and

(x) $\mathrm{ind}(T_J) \geq 8$

is due to [KMRT, (40.7)]. Here condition (x) fits into 4.4 as follows. Since S_J and T_J up to a sign agree on linear hyperplane of J (by (1.4.4)), their Witt indices differ by at most 1. Hence (iv) implies (x) and (x) implies (v), showing without recourse to loc. cit. that (i) - (x) are equivalent. $\square$

4.6 Isotopy versus isomorphism of Albert division algebras. While it is easy to construct reduced Albert algebras which are isotopic but not isomorphic, the case of Albert division algebras is more difficult. A unified solution to this problem, for whose geometric significance we refer to Tits-Weiss [TW, §38], will be given in the following application of 4.4.

4.7 Theorem. (cf. [TW, (38.9)]) *Notations being as in* 4.0, *let $E \subset J$ be a cubic étale subalgebra. Then there exists a $v \in E^\times$ such that $\pi_5(J^{(v)})$ is hyperbolic. In particular, if $\pi_5(J)$ is anisotropic, $J^{(v)}$ cannot be isomorphic to J.*

Proof. We may assume that k is infinite. Then Zariski density produces an element $y \in E^\perp$ satisfying the hypotheses of 1.16. Hence the subalgebra of J generated by E and y, as it arises from the étale Tits process (1.16), has the form $H(B, \tau)$ for some central simple associative k-algebra (B, τ) of degree 3 with involution of the second kind. By 2.10, some $v \in E^\times$ makes $\mathrm{Int}(v) \circ \tau$ a distinguished involution, and 4.4 implies that $\pi_5(J^{(v)})$ is hyperbolic. $\square$

4.8 Example: Generic matrices. Let J be the Albert algebra of generic matrices of k [P7]. Thus J is an Albert division algebra over some extension field of k [P7, Theorem 1] and, given any separable cubic subfield $E \subset J$, S_J is anisotropic on $E^\perp$ [P7, Theorem 2]. Hence $\pi_5(J)$ is anisotropic as well (4.4), so E contains an invertible element v such that J and $J^{(v)}$ are not isomorphic (4.7). $\square$

We now pass to the 3-invariant mod 2 of J. The following technical result generalizes [PR8, 4.12]. For simplicity, we confine ourselves to division algebras.

4.9 Lemma. *Let (B, τ) be a central simple associative k-algebra of degree* 3 *with involution of the second kind and $b \in K = \mathrm{Cent}(B)$ satisfy $N_K(b) = 1$. Assume that $J = J(B, \tau, 1, b)$ is a division algebra and let $E \subset H(B, \tau)$ be a separable cubic subfield. Writing J' for the subalgebra of J generated by E and $y = (0, 1) \in J$, there exists an element $b' \in K$ satisfying $N_K(b') = 1$ and $J' \cong J(E, K, 1, b')$.*

Proof. J being a division algebra, we obtain $K = k[b]$ since, otherwise, $b = \pm 1$ would be a generic norm of B. Also, by (1.9.1), (1.9.2), $T_K(b) = N_J(y) \neq 0$ and $y^\sharp = (-1, \tau(b))$. Thus, since J' is a division algebra of degree 3 and dimension 9, we may apply 1.16 to conclude $J' \cong J(E, L, 1, b')$, where $L = k[c]$ is the quadratic étale k-algebra generated by an element c with minimum polynomial

$X^2 - X + T_K(b)^{-2}$ over k and $b' \in L$ has norm 1. Hence b and $T_K(b)c$ have the same minimum polynomial, and the proof is complete. □

4.10 Theorem. *Notation being as in* 4.0, *let* $E \subset J$ *be a cubic étale subalgebra. Then the following statements are equivalent.*

(i) $\pi_3(J)$ *is hyperbolic.*

(ii) Q_J *is hyperbolic.*

(iii) $\mathrm{ind}(Q_J) \geq 9$.

(iv) S_J *has maximal Witt index for* $\mathrm{char}\, k \neq 2$, *and* $\mathrm{ind}(S_J^0) \geq 12$ *for* $\mathrm{char}\, k = 2$.

(v) $\mathrm{ind}(S_J^*) \geq 11$.

(vi) $\mathrm{ind}(S_J|_{E^\perp}) \geq 11$.

(vii) $\mathrm{ind}(S_J|_{E^\perp}) \geq 10$.

(viii) J_{red} *is split.*

(ix) *Every reducing field of* J *splits* J.

(x) J *is a first Tits construction.*

Proof. (i) $\Longrightarrow$ (ii). $\pi_5(J) \cong \pi_3(J) \perp \langle -1 \rangle \,.\, Q_J$ (by (4.1.1), (4.1.2)), being a multiple of $\pi_3(J)$, must be hyperbolic as well, giving (ii).
the implications (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v), (vi) $\Rightarrow$ (vii) are clear.
(iii) $\Longrightarrow$ (i). Combining (iii) with 4.4 we conclude that J_{red} contains nonzero nilpotent elements and hence has the form $J_{\mathrm{red}} \cong H_3(C, g)$, where C is the co-ordinate algebra of J and $g = \mathrm{diag}(1, -1, 1)$ (1.8). This implies $Q_J = Q_{J_{\mathrm{red}}} = \langle 1, -1, 1 \rangle \,.\, \pi_3(J)$ (by (1.7.7)) $= \pi_3(J)$ in the Witt group of k, and (i) follows.
(ii) $\Longrightarrow$ (iv). This follows from (2.1.3), (2.1.1).
(v) $\Longrightarrow$ (iii). This follows from (2.1.3) by counting dimensions of totally isotropic subspaces.
(ii) $\Longrightarrow$ (vi). This follows from 4.2 by counting dimensions of totally isotropic subspaces.
(vii) $\Longrightarrow$ (iii). Dito.
(i) $\Longleftrightarrow$ (viii). This follows from $\pi_3(J) \cong \pi_3(J_{\mathrm{red}})$.
(viii) $\Longrightarrow$ (ix). Every reducing field E of J satisfies $J \otimes E \cong J_{\mathrm{red}} \otimes E$ (by 1.17) and this is split.
(ix) $\Longrightarrow$ (viii). There exists a field extension of degree 1 or 3 which splits J, hence J_{red}. By Springer's Theorem, J_{red} must have been split to begin with.
(x) $\Longrightarrow$ (ix). This is [PR1, Corollary 4.2].
(ix) $\Longrightarrow$ (x). We may assume that J is a division algebra. By [PR1, Theorem 4.8] all isotopes of J are isomorphic, so J can be obtained by the unital Tits Process [PR7, 4.2]: There exist a central simple associative algebra (B, τ) of degree 3 with involution of the second kind as well as an element $b \in K = \mathrm{Cent}(B)$ satisfying $N_K(b) = 1$ and $J \cong J(B, \tau, 1, b)$. By 4.1, and since (i) holds, τ is distinguished. Hence 3.1 yields a separable cubic subfield $E \subset H(B, \tau)$ satisfying $\delta(E/k) = \delta(K/k)$, and 4.9 yields an element $b' \in K$ having norm 1 such that the unital étale Tits process $J' = J(E, K, 1, b')$ becomes a subalgebra of J. But by 1.14, J' has the form D^+ for some central associative division algebra D of degree 3 over k, forcing J to be a first Tits construction.
□

4.11 Remark. a) Assume char $k \neq 2$. Arguing as in 4.5, we see that (i) - (x) of 4.9 are equivalent to

(xi) $\mathrm{ind}(T) \geq 12$,

thus recovering most of [KMRT, (40.5)].
b) Assume char $k = 2$. Then (2.1.2), (2.1.3) show that, if (i) - (x) of 4.9 hold, S_J^0 has maximal Witt index if and only if k contains the cube roots of 1. Thus 4.9 generalizes [PR1, Theorem 4.7] (see [KMRT, (40.6)]).

5. A Skolem-Noether theorem for Albert algebras.

5.0 We now proceed to derive a Skolem-Noether type theorem, due to Parimala-Sridharan-Thakur [PaST] over base fields of characteristic not 2 or 3, in full generality. To this end, we fix a central simple associative k-algebra (B, τ) of degree 3 with involution of the second kind and write $K = \mathrm{Cent}(B)$ for the centre of B.

Besides the invariants mod 2 of an Albert algebra J, its invariant mod 3, denoted by $g_3(J)$, also plays a central role in our subsequent investigation. We refer to [KMRT, p. 537] for the definition of this invariant and to Rost [Ro] for an existence proof (provided char $k \neq 2, 3$). An elementary approach valid in all characteristics has been described by Petersson-Racine [PR8], [PR9]. In particular, we always know that J is a division algebra if and only if $g_3(J) \neq 0$. Hence a standard argument, reproduced in [PR5, p. 204] or [KMRT, proof of (40.10)], for example, leads to the following general conclusion.

5.1 Proposition. *Let A be a central simple associative algebra of degree 3 over k. Then, given $\alpha, \alpha' \in k^\times$, the following statements are equivalent.*

(i) *$J(A, \alpha')$ and $J(A, \alpha)$ are isomorphic.*

(ii) *$g_3(J(A, \alpha')) = g_3(J(A, \alpha))$.*

(iii) *There exists $w \in A^\times$ satisfying $\alpha' = N_A(w)\alpha$.*

(iv) *The identity of A extends to an isomorphism from $J(A, \alpha')$ onto $J(A, \alpha)$.* □

Actually, the aforementioned theorem of Parimala-Sridharan-Thakur (which we shall now represent in full generality) specializes to 5.1 if K is split, forcing (B, τ) to have the form $(A \times A^{\mathrm{op}}, \varepsilon)$ for some central simple associative algebra A of degree 3, ε being the exchange involution.

5.2 Theorem. *Notations being as in 5.0, let $u, u' \in H(B, \tau), b, b' \in K$ be invertible elements satisfying $N_B(u) = N_K(b)$, $N_B(u') = N_K(b')$. Then, setting $J = J(B, \tau, u, b)$, $J' = J(B, \tau, u', b')$, the following statements are equivalent.*

(i) *J' and J are isomorphic.*

(ii) *J' and J are isotopic.*

(iii) *$\pi_3(J') \cong \pi_3(J)$ and $g_3(J') = g_3(J)$.*

(iv) *There exists $w \in B^\times$ satisfying $u' = wu\tau(w)$, $b' = N_B(w)b$.*

(v) *The identity of $H(B,\tau)$ extends to an isomorphism from J' onto J.*

5.3 Proof of 5.2, Part I. The implications (v) $\Rightarrow$ (i) $\Rightarrow$ (ii) are obvious. For the implication (ii) $\Rightarrow$ (iii) we refer to Thakur [Th, Theorem 2.2], where the restrictions on the characteristic are actually unnecessary. Finally, since the implication (iv) $\Rightarrow$ (v) immediately follows from 1.10, it remains to show that (iii) implies (iv). To do so, we require a preparation, generalizing [KMRT, (40.13)] to base fields of arbitrary characteristic.

5.4 Lemma. *Let $w \in B^\times$ and suppose $\lambda = N_B(w) \in K^\times$ satisfies $N_K(\lambda) = 1$. Then there exists an element $w' \in B^\times$ satisfying $\lambda = N_B(w')$ and $w'\tau(w') = 1$.*

Proof. We adapt the proof of [PT, 4.5] to the present set-up and first assume that $J_1 = H(B,\tau)$ is reduced, having the form $H_3(K,g)$ for some diagonal matrix $g \in \mathrm{GL}_3(k)$. This implies $B = M_3(K)$, and $w' = \mathrm{diag}(1,1,\lambda) \in B$ does the job. We are left with the case that J_1 is a division algebra. Choosing $\mu \in K^\times$ such that $\lambda = \mu\tau(\mu)^{-1}$, we pick $\theta \in K - k$ satisfying $T_K(\theta) = 1$ and $\kappa \in K^\times$ satisfying $\tau(\kappa) = -\kappa$ to define a cubic form $F : J_1 \times k \longrightarrow k$ by

$$F\big((x,\xi)\big) := \kappa\left[\tau(\mu)N_B(x + \xi\theta 1) - \mu N_B(x + \xi(1-\theta)1)\right]$$

for $x \in J_1$, $\xi \in k$. Then we distinguish the following cases.

Case 1. $K \cong k \times k$ splits.
Then $(B,\tau) \cong (A \times A^{\mathrm{op}}, \varepsilon)$ where A is a central simple associative k-algebra of degree 3 and ε stands for the exchange involution on $A \times A^{\mathrm{op}}$, forcing $\lambda = (\alpha, \alpha^{-1})$, $\alpha = N_A(w_1)$ for some $w_1 \in A^\times$. Hence $w' := (w_1, w_1^{-1})$ does the job. We also claim that F is isotropic. To see this, we write $\theta = (\beta, 1-\beta)$, $\beta \in k$, $2\beta \neq 1$ and may assume $\mu = (\alpha, 1)$ as well as $w_1 \neq 1$ (otherwise $F\big((1,0)\big) = 0$). Setting $u = (w_1 - 1)^{-1}\big(\beta 1 + (\beta - 1)w_1\big) \in A$ and $x = (u,u) \in J_1$, a routine computation gives $F\big((x,1)\big) = 0$.

Case 2. K is a field.
We may assume $\lambda \neq 1$. Since F, by the discussion of Case 1, becomes isotropic after extending scalars from k to K, it must have been so all along [SV, 4.2.11]. This yields a nonzero element $(x,\xi) \in J_1 \times k$ such that

$$\tau(\mu)N_B(x + \xi\theta 1) = \mu N_B\big(x + \xi(1-\theta)1\big) = \mu N_B\big(\tau(x + \xi\theta)\big).$$

Hence $x + \xi\theta 1 \in B^\times$, and $w' = (x + \xi\theta 1)\tau(x + \xi\theta 1)^{-1}$ satisfies $\lambda = N_B(w')$ as well as $w'\tau(w') = 1$, since the factors of w' belong to the K-algebra generated by x and hence commute. This completes the proof. $\square$

5.5 Proof of 5.2, Part II. We can now settle the sole remaining implication (iii) $\Rightarrow$ (iv) of 5.2. If $K \cong k \times k$ splits, J and J' are first Tits constructions, so 5.1 combines with 1.12 to give (iv). Hence we may assume that K is a field. Changing scalars from k to K transforms J and J' into the first Tits constructions $J(B,b)$, $J(B,b')$, respectively, having the same invariant mod 3. This implies $b' = N_B(w_1)b$ for some $w_1 \in B^\times$ (5.1), allowing us to assume $b = b'$ (1.10). On the other hand, we conclude from 4.1 that $\pi_3(J)$, $\pi_3(J')$ are the Pfister forms of the involutions $\tau^{(u)} = \mathrm{Int}(u) \circ \tau$, $\tau^{(u')} = \mathrm{Int}(u') \circ \tau$, respectively, on B. Since $\pi_3(J) \cong \pi_3(J')$ by (iii), we may apply 2.4 to find an invertible element $v \in B$ satisfying $\tau^{(u')} = \mathrm{Int}(v) \circ \tau^{(u)} \circ \mathrm{Int}(v)^{-1}$. This implies $u' = \alpha v u \tau(v)$ for some $\alpha \in k^\times$. Using the relations $N_B(u) = N_K(b) = N_B(u')$, we deduce $\alpha^3 N_B(v)\tau\big(N_B(v)\big) = 1$, so $v_1 = \alpha^2 N_B(v)v \in B^\times$ satisfies $v_1 u \tau(v_1) = u'$. Performing the same computations again, with $v_1, 1$ in place of v, α, respectively, we see that $\lambda = N_B(v_1)$ satisfies $\lambda\tau(\lambda) = 1$. Applying 5.4 to λ^{-1} and $\tau^{(u')}$ yields an element $v_2 \in B^\times$ such that $N_B(v_2) = \lambda^{-1}$, $v_2 \tau^{(u')}(v_2) = 1$. The latter amounts to $u' = v_2 u' \tau(v_2)$, so $w = v_2 v_1$ satisfies all requirements of (iv) in 5.2. $\square$

5.6 Remark. a) In the spirit of [Th, Theorem 2.1] and its proof, 5.2 generalizes easily to the situation where two distinct involutions of the second kind (rather than τ alone) are allowed on B; no restrictions on the characteristic have to be imposed.
b) Just as in [KMRT, (40.15)] or [PaST, Section 3] one may use 5.1, 5.2 to establish the classical Skolem-Noether theorem for 9-dimensional separable subalgebras of Albert algebras in arbitrary characteristic.

6. The Tits process and Pfister forms

6.0 As before, we let (B,τ) be a central simple associative algebra of degree 3 over k with involution of the second kind and write $K = \mathrm{Cent}(B)$ for the centre of B. In 3.3 we have described the Pfister form of $J = H(B,\tau)$ explicitly in terms of parameters needed to realize J by means of the étale Tits process. On the other hand, for char $k \neq 2$, [KMRT, (19.25)] provides a similar description in terms of K and an arbitrary cubic étale k-subalgebra of J. It is the purpose of the present section to compare these two descriptions. In doing so, we will obtain a version of [KMRT, (19.25)] that is valid in all characteristics. Our approach is based on a number of elementary computations in the Witt group of k.

6.1 Lemma. *Let E be a cubic étale k-algebra and $u \in E$ an invertible element. Then*

$$(T_E)_*(\langle u \rangle) \,.\, N_{\Delta(E)} = \langle N_E(u) \rangle \,.\, N_{\Delta(E)}$$

in the Witt group of k.

Proof. By Springer's theorem, we may assume $E = k \times \Delta$, $\Delta = \Delta(E)$. Then $u = (\alpha, v)$ ($\alpha \in k^\times$, $v \in \Delta^\times$), forcing $(T_E)_*(\langle u \rangle) = \langle \alpha \rangle \perp (T_\Delta)_*(\langle v \rangle)$, hence

$$(T_E)_*(\langle u \rangle) \,.\, N_\Delta = \langle \alpha \rangle \,.\, N_\Delta + (T_\Delta)_*(\langle v \rangle) \,.\, N_\Delta = \langle \alpha \rangle \,.\, N_\Delta$$

in the Witt group of k since $(T_\Delta)_*(\langle v \rangle) \,.\, N_\Delta = (T_\Delta)_*\big(\langle v \rangle . (N_\Delta \otimes \Delta)\big)$ (by Frobenius reciprocity (1.6.1)) is hyperbolic. On the other hand, $N_E(u) = \alpha N_\Delta(v)$ implies

$$\langle \alpha \rangle \,.\, N_\Delta = \langle N_E(u) \rangle \langle N_\Delta(v) \rangle \,.\, N_\Delta = \langle N_E(u) \rangle \,.\, N_\Delta.$$

$\square$

6.2 Lemma. *Notations being as in* 6.0, *let $E \subset J$ be any cubic étale subalgebra and write L for the quadratic étale k-algebra corresponding to $\delta(K/k) + \delta(E/k) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. Then*

$$\pi(J) \cong \langle -d_{K/k} \rangle \,.\, \pi(J) \cong \langle -d_{L/k} \rangle .\, \pi(J) \cong \langle d_{E/k} \rangle \,.\, \pi(J).$$

Proof. Since $\pi(J)$ is a multiple of N_K (by (2.2.3)) and of N_L as well (by 3.3), the first two relations follow from 1.3.4. Now 1.14 implies the rest. $\square$

6.3 Lemma. *Notations being as in* 6.2,

$$N_L = N_{\Delta(E)} + \langle d_{E/k} \rangle \,.\, N_K$$

in the Witt group of k.

Proof. The idea is to compute the $\delta(E/k)$-associate of $N_K \perp \mathbf{h}$ in two different ways. Using (1.3.1), (1.3.2), we obtain

$$(N_K \perp \mathbf{h})_{\delta(E/k)} \cong (N_K)_{\delta(E/K)} \perp \mathbf{h} \cong N_L \perp \mathbf{h}$$

on the one hand and

$$(N_K \perp \mathbf{h})_{\delta(E/k)} \cong (N_{k\{0\}} \perp N_K)_{\delta(E/k)} \cong N_{\Delta(E)} \perp \langle d_{E/k} \rangle .\, N_K$$

on the other. Comparing the two leads to the desired conclusion. $\square$

6.4 Normalizing the Tits process. Let K be a quadratic étale k-algebra, B a separable associative K-algebra of degree 3, τ a K/k-involution of B and $u \in H(B, \tau)$, $b \in K$ invertible elements satisfying $N_B(u) = N_K(b)$. Just as in [KMRT, (39.2)] we may modify the Tits process $J(K, B, \tau, u, b)$ by setting $w := b^{-1}u$, $v := wu\tau(w)$, $c := bN_B(w)$ to obtain

$$v = N_B(u)^{-1}u^3, \; c = \tau(b)b^{-1}$$

and an isomorphism $J(K, B, \tau, u, b) \xrightarrow{\sim} J(K, B, \tau, v, c)$ via $(x_0, x) \mapsto (x_0, xw)$ (1.10), where the new Tits process is *normalized* in the sense that $N_B(v) = N_K(c) = 1$.

6.5 Theorem. (cf. [KMRT, (19.25)]) *Notations being as in* 6.0, *let $E \subset J$ be any cubic étale subalgebra. Realizing $J \cong J(E, L, u, b)$ as an étale Tits process algebra* (cf. 1.14), *where L is a quadratic étale k-algebra and $u \in E$, $b \in L$ are invertible elements satisfying $N_E(u) = N_L(b) \in k^{\times 2}$* (cf. 6.4), *we obtain*

$$\pi(J) \cong \big(\langle 1 \rangle \perp \langle d_{E/k} \rangle (T_E)_*(\langle u \rangle)\big) \,.\, N_K.$$

Proof. Since both sides have the same dimension, its suffices to show that they determine the same element in the Witt group of k. Accordingly, we compute

$$\begin{aligned}
\pi(J) &= N_L + \langle d_{E/k}\rangle (T_E)_*(\langle u\rangle)\,.\,N_L && \text{(by 3.3)}\\
&= N_{\Delta(E)} + \langle d_{E/k}\rangle\,.\,N_K + \\
&\quad \langle d_{E/k}\rangle (T_E)_*(\langle u\rangle)\,.\,N_{\Delta(E)} + (T_E)_*(\langle u\rangle)\,.\,N_K && \text{(by 6.3)}\\
&= N_{\Delta(E)} + \langle d_{E/k}\rangle\,.\,N_K + \langle d_{E/k}\rangle\,.\,N_{\Delta(E)} + \\
&\quad (T_E)_*(\langle u\rangle)\,.\,N_K && \text{(by 6.1 and } N_E(u)\in k^{\times 2})\\
&= \langle d_{E/k}\rangle\,.\,\big(N_K + \langle d_{E/k}\rangle (T_E)_*(\langle u\rangle)\,.\,N_K\big) && \text{(by (1.3.4)).}
\end{aligned}$$

Since $\pi(J)\cong\ \langle d_{E/k}\rangle\,.\,\pi(J)$ by 6.2, the assertion follows. □

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