VALUATIONS ON COMPOSITION ALGEBRAS

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Abstract

Necessary and sufficient conditions for a valuation on a field to extend to a central simple nonassociative algebra of finite dimension are obtained. Applications are given to valuations of composition algebras; in particular, we describe all quaternion algebras over the rationals to which the p–adic valuation, p a prime, may be extended.

Non-archimedian valuations on composition algebras over the field $\mathbf{Q}$ of rational numbers have recently been studied in Balachandran, Rema and Satyanarayanamurthy [1]. It is the purpose of the present note to

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propose a different approach to the subject, leading at the same time to more general results and to shorter proofs as well.

1. Let k be a field. All k-algebras are supposed to be nonassociative with unit. By a (multiplicative) *valuation* on a k-algebra A we mean a map $v, x \mapsto |x|$, from A to the nonnegative reals satisfying the usual conditions

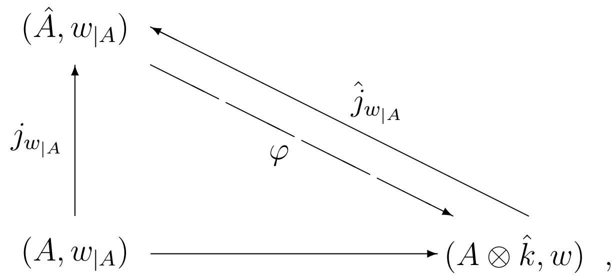
$$\begin{aligned} &|x| = 0 \Longleftrightarrow x = 0, \\ &|x+y| \le |x| + |y| , \\ &|xy| = |x|\,|y| \end{aligned}$$

for all $x, y \in A$. For $A \neq 0$ to admit a valuation it is clearly necessary that it contain no divisors of zero (hence, in the finite-dimensional case, that it be a division algebra). Every valuation of A canonically induces a valuation on k.

2. Conversely, let k be a valued field with completion $\hat{k}$ and suppose A is a central simple k-algebra of finite dimension. We wish to set up a bijection between the valuations of A inducing the given valuation on k and the valuations of $A \otimes_k \hat{k}$ inducing the given valuation on $\hat{k}$. On the one hand, if we start with a valuation w of $A \otimes \hat{k}$ inducing the given valuation on $\hat{k}$, restriction yields a valuation $w_{|A}$ of A inducing the given valuation on k. On the other hand, if we start with a valuation v of A inducing the given valuation on k and denote by $j_v : A \to \hat{A}$ the canonical imbedding into the corresponding completion, the composite map $k \to A \to \hat{A}$ extends to a homomorphism $\hat{k} \to \hat{A}$, giving $\hat{A}$ the structure of a $\hat{k}$-algebra. Hence j_v extends to a homomorphism $\hat{j}_v : A \otimes \hat{k} \to \hat{A}$, which, by central simplicity, is injective and so may be used to pull back the valuation of $\hat{A}$ to yield a valuation $v \otimes \hat{k}$ of $A \otimes \hat{k}$ inducing the given valuation on $\hat{k}$.

3. Proposition *Let k be a valued field with completion $\hat{k}$ and A a central simple k-algebra of finite dimension. Then the assignments $v \mapsto v \otimes \hat{k}$, $w \mapsto w_{|A}$ defined in* **2**. *yield inverse bijections between the set of valuations of A inducing the given valuation on k and the set of valuations of $A \otimes_k \hat{k}$ inducing the given valuation on $\hat{k}$.*

Proof. The relation $(v \otimes \hat{k})_{|A} = v$ being straight forward to check, it remains to prove $w_{|A} \otimes \hat{k} = w$. Since A has finite dimension, w is complete, so the natural map $A \to A \otimes \hat{k}$ extends to a homomorphism φ preserving valuations, as shown in the diagram



$(\hat{A}, w_{|A})$ being the completion of $(A, w_{|A})$. Hence $\hat{j}_{w_{|A}}$ is an isomorphism with inverse φ, and the assertion follows.

Proposition 3 has the following immediate application, generalizing [1, Lemma 3.1].

4. Corollary *Let k be a valued field with completion $\hat{k}$ and C a central composition algebra over k with norm n . Then the valuation of k extends to a valuation of C if and only if $C \otimes_k \hat{k}$ is a division algebra* [1]*. In this case, such an extension is necessarily unique, given by the formula $x \mapsto |n(x)|^{\frac{1}{2}}$, and preserves automorphisms as well as anti-automorphisms of C ; in particular, it preserves conjugation. It is respectively archimedean, non-archimedean, discrete if and only if the valuation of k has the corresponding property .*

Proof. Proposition 3 allows us to assume, whenever necessary, that k is complete and so is non-archimedean or agrees with $\mathbf{R}$ or $\mathbf{C}$. In any event, we may either pass to the quadratic Jordan algebra associated with C and invoke the valuation theory developed in [5], particularly Satz 5.1 and the subsequent Bemerkung 2, or use a well known theorem of Springer [7] to conclude the proof.

5. Remark (i) Granting the obvious adjustments of Proposition 3 to

[1] The author is indebted to W. Scharlau, who suggested this simple but important result during a bycicle ride to the mathematics department of the University of Münster in the fall of 1971.

the Jordan setting, the argument given above carries over directly to finite-dimensional absolutely simple quadratic Jordan algebras, producing a result which is completely analogous to Corollary 4 and contains the corresponding statement for finite-dimensional central simple associative algebras as a special case. We omit the details.
(ii) In [1] only valuations preserving conjugation were considered. According to Corollary 4, this restriction is a vacuous one.
(iii) It is not difficult to obtain a version of Proposition 3 for non-associative algebras which may not contain a unit: Indeed, given a non-zero k-Algebra A, not necessarily unital, a valuation $x \mapsto |x|$ on A is easily seen to induce a unique valuation $\alpha \mapsto |\alpha|$ on k satisfying $|\alpha x| = |\alpha||x|$ for all $\alpha \in k$ and all $x \in A$. Now **2.**, **3.** carry over verbatim to the more general setting of non-unital algebras, the concept of central simplicity being understood in the sense of Jacobson [3,X §1].
(iv) The fact that there exists at most one extension of the valuation of k to a valuation of C is originally due to Eichhorn [2, Satz 10].
(v) The following statement generalizes a result announced in [1].

6. Corollary *Let C be an octonion algebra over a number field k. Then a non-real valuation v on k does not extend to a valuation of C.*

PROOF. Indeed, $\hat{k}$ being either the field of complex numbers or a local field with finite residue field, it is a standard fact that there are no octonion division algebras over $\hat{k}$. Hence Corollary 4 applies.

7. We now turn to a question that has been discussed in [1] at length: Given a prime number p and a quaternion algebra D over $\mathbf{Q}$, what does it mean that the p-adic valuation of $\mathbf{Q}$ extends to a valuation of D ? Below we will give a quick answer to this question by using Corollary 4 and the theory of local symbols as developed in Serre [6, Chap. XIV] [2]. Adopting the usual notation, we let (r, s), for non-zero rational numbers r, s, be the rational quaternion algebra with norm

$$\langle 1, -r, -s, rs \rangle = x^2 - ry^2 - sz^2 + rsw^2.$$

(This seems to agree with the algebra $D(-r, -s)$ in [1].) On the other

[2]The author, who originally had proceeded in a slightly different manner, is indebted to W. Scharlau and M. Schulte for having drawn his attention to this.

hand, we have the p-adic symbol $(r,s)_p \in \{\pm 1\}$ [6, XIV § 2, p. 215, with $n = 2$], which is -1 if $\mathbf{Q}_p$ does not split the quaternion algebra (r,s) (i.e., by Corollary 4, if the p-adic valuation of $\mathbf{Q}$ extends to a valuation of (r,s)) and 1 otherwise ([6, XIV Proposition 7] and [3, 57:9]). Similar to [1], we may assume, whenever necessary, that

$$D = (m,n) \quad \text{or} \quad D = (m,pn) \quad \text{or} \quad D = (pm,pn),$$

where m,n are integers not divisible by p. Moreover, since $(m,pn) \tilde{=} (pn,m) \tilde{=} (pn,-pmn)$ and $(pm,pn) \tilde{=} (pm,-p^2mn) \tilde{=} (-mn,pm)$ by [3, 57:10], the second case may always be translated to the third and conversely.

8. Suppose now that p is odd. Writing non-zero integers m,n as

$$m = p^\alpha m', \quad n = p^\beta n',$$

with $\alpha,\beta \in \mathbf{Z}$ non-negative and $m',n' \in \mathbf{Z}$ not divisible by p, we can express $(m,n)_p$ via

$$(m,n)_p = (-1)^{\alpha\beta\,\frac{p-1}{2}} \, (\frac{n'}{p})^\alpha \, (\frac{m'}{p})^\beta$$

as a product of Legendre symbols [6, Chap. XIV § 4, p. 218]. Combining this with **7.**, we conclude

9. Corollary [1, Theorems 3.7, 3.11] *Let p be an odd prime and m,n integers not divisible by p. Then the p-adic valuation of $\mathbf{Q}$ does not extend to a valuation of the quaternion algebra (m,n). It extends to a valuation of the quaternion algebra (pm,pn) if and only if*

$$\left(\frac{mn}{p}\right) = (-1)^{\frac{p+1}{2}}.$$

10. We are left with the case $p = 2$. Fixing odd integers m,n, we have

$$(m,n)_2 = (-1)^{\frac{m-1}{2}\,\frac{n-1}{2}}$$

by [loc. cit., p. 219] and conclude

11. Corollary [1, Theorem 3.5] *Let m,n be odd integers. Then the*

2-adic valuation of $\mathbf{Q}$ *extends to a valuation of the quaternion algebra* (m, n) *if and only if* $m \equiv n \equiv 3 \bmod 4$.

Since the 2-adic symbol may be viewed as a symmetric bilinear form on the $\mathbf{F}_2$-vectorspace $\mathbf{Q}_2^\times / \mathbf{Q}_2^{\times 2} (\mathbf{Q}_2^\times = \mathbf{Q}_2 - \{0\})$, we finally obtain, $m, n \in \mathbf{Z}$ still being odd,

$$(m, 2n)_2 = (m, 2)_2 (m, n)_2 = (-1)^{\frac{m^2-1}{8}} (-1)^{\frac{m-1}{2}\frac{n-1}{2}}$$

from [loc. cit.], whence the 2-adic valuation of $\mathbf{Q}$ extends to valuation $(m, 2n)$ if and only if either $m \equiv 3, 5 \bmod 8$ or $m \equiv n \equiv 3 \bmod 4$ (but not both). We thus end up with the following result.

12. Corollary *Let* m, n *be odd integers. Then the* 2*-adic valuation of* $\mathbf{Q}$ *extends to a valuation of the quaternion algebra* $(m, 2n)$ *if and only if one of the following conditions is fulfilled.*

(i) $m \equiv 3 \bmod 8$ *and* $n \equiv 1 \bmod 4$.

(ii) $m \equiv 5 \bmod 8$.

(iii) $m \equiv 7 \bmod 8$ *and* $n \equiv 3 \bmod 4$.

Since $(2m, 2n) \cong (2m, -4mn) \cong (-mn, 2m)$ by [3, 57:10], Corollary 12 agrees with the theorem stated without proof in [1, p. 118].

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