The structure group of an alternative algebra

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Abstract

The structure group of an alternative algebra and various canonical subgroups are defined and investigated. Using the principle of triality, natural sets of generators for these groups in the case of octonion algebras are exhibited.

The structure group of a Jordan algebra may be defined as the group of isomorphisms from the algebra onto its various isotopes. Since the notion of isotopy, thanks to the work of McCrimmon [9], extends naturally to the setting of alternative algebras, so does the notion of the structure group. However, there is an important difference in this context. While isotopes of a Jordan algebra depend on a *single* invertible element, which happens to be unique, isotopes of an alternative algebra depend on a *pair* of invertible elements, which is unique only up to a multiplicative shift by invertible elements in the nucleus.

In the present paper we take this difference into account by defining the structure group of an alternative algebra A as the totality of triples (η, u, v) where η is an isomorphism from A onto its (u, v)-isotope and $u, v \in A$ are both invertible. This seems to be a much more natural object to study than the *narrow* structure group of A, consisting by definition of isomorphisms from A onto appropriate (but unspecified) isotopes. For example, the structure group can always be regarded as an affine group scheme in a natural way, whereas the narrow structure group apparently cannot. On the other hand, the structure group is easily seen to be an extension in the group-theoretical sense of the narrow structure group by the unit group of the nucleus. This extension turns out to be split if the unterlying algebra is associative; at the other extreme, e.g., for an octonion algebra over a field which is algebraically closed, the extension is not split when interpreted in the category of *algebraic* groups. We also show that the structure group can be thought of as the group of Albert autotopies in disguise [1], thereby providing a realization of the latter which is intimately tied up with the alternative structure of the underlying algebra. Furthermore, elements of the structure group play an important role in Albert's classical approach [2] to the construction of Albert division algebras, and in Loos' treatment [7] of alternative pairs with invertible elements.

Finally, the structure group is shown to be closely connected with a group of transformations that has recently been studied by Tits and Weiss [16] in their classification theory of Moufang polygons. We use this connection and the principle of triality to exhibit natural sets of generators both for the structure group and for the group of Tits and Weiss when the underlying algebra is an octonion algebra. The main constituents for these generators are inner structural transformations, which generalize inner automorphisms to the setting of alternative algebras, and $*$-close sequences, where $*$ stands for the canonical involution of the octonion algebra and a finite sequence $(v_1, \dots, v_r)$ of invertible elements is said to be $*$-*close* if it satisfies the relation

$$v_1\big(v_2(\dots(v_{r-1}v_r)\dots)\big) = \pm v_1^*\big(v_2^*(\dots(v_{r-1}^*v_r^*)\dots)\big).$$

We establish and use a version of Hilbert's Theorem 90 for composition algebras to prove that, roughly speaking, $*$-close sequences of arbitrary length ≥ 3 exist in abundance.

1. Isotopes of alternative algebras

1.0 Generalities. Let k be a commutative associative ring of scalars. Unless explicitly stated otherwise, all (nonassociative) algebras considered in the sequel are assumed to be over k and to contain an identity element 1. Subalgebras are always unital, and algebra homomorphisms always take 1 into 1. Left, right multiplication by an element x in an algebra A will be written als L_x, R_x, respectively. The nucleus of A, defined to be the set of all elements in A which associate with everything else in A, will be denoted by $\mathrm{Nuc}(A)$; it is an associative subalgebra. We write $\mathrm{Cent}(A)$ for the centre and A^{op} for the opposite algebra of A.

1.1 Quadratic Jordan algebras. We will occasionally employ rudiments from the theory of (unital) quadratic Jordan algebras. The standard reference is Jacobson [3]. Recall that a (quadratic) Jordan algebra consists of a k-module J, a distinguished element $1 \in J$ (the *unit*) and a quadratic map $U : J \longrightarrow \mathrm{End}_k(J)$ (the U-*operator*) such that the relations

$$U_1 = \mathbf{1}_J, \tag{1.1.1}$$

$$U_{U_x y} = U_x U_y U_x, \qquad \text{(“fundamental formula”)} \tag{1.1.2}$$

$$U_x V_{y,x} = V_{x,y} U_x \tag{1.1.3}$$

hold under all scalar extensions where

$$V_{x,y} z := U_{x,z} y := [U_{x+z} - U_x - U_z]y. \tag{1.1.4}$$

$x \in J$ is said to be *invertible* if $U_x : J \longrightarrow J$ is bijective, in which case we put $x^{-1} := U_x^{-1} x$. The set of invertible elements in J will be denoted by $J^\times$. For $y \in J^\times$, the new unit $1^{(y)}$ and the new U-operator $U^{(y)}$ defined by

$$1^{(y)} := y^{-1}, \; U_x^{(y)} := U_x U_y \tag{1.1.5}$$

give the k-module J a new Jordan algebra structure, denoted by $J^{(y)}$ and called the y-*isotope* of J. We have $J^{(y)\times} = J^\times$, and the inverse of $x \in J^\times$ in $J^{(y)}$ is

$$x^{(-1,y)} = U_{y^{-1}} x^{-1}. \tag{1.1.6}$$

The *structure group of* J, denoted by $\mathrm{Str}(J)$, is the subgroup of the full linear group of J consisting of all isomorphisms η from J onto the isotope $J^{(y)}$ of J, for some $y \in J^\times$ depending on η. In fact, y can be recovered from η via the formula $y^{-1} = 1^{(y)}$ (by 1.1.5) $= \eta(1)$. The automorphism group of J agrees with the stabilizer group of 1 in $\mathrm{Str}(J)$.

1.2 Alternative algebras. Let C be an alternative algebra, so C is flexible,

$$(xy)x = x(yx) =: xyx, \tag{1.2.1}$$

and satisfies the alternative laws,

$$x(xy) = x^2 y, \; (yx)x = yx^2, \tag{1.2.2}$$

as well as Moufang's identities

$$x(y(xz)) = (xyx)z, ((zx)y)x = z(xyx), \tag{1.2.3}$$

$$(xy)(zx) = x(yz)x. \tag{1.2.4}$$

A standard reference for alternative algebras is Schafer [13]. Repeated use will be made of Artin's Theorem, which says that alternative algebras on two generators are associative. The k-module C together with the unit 1 and the U-operator defined by

$$U_x y = xyx \tag{1.2.5}$$

is a Jordan algebra denoted by C^+.

1.3 Invertibility. Following McCrimmon [9], an element $u \in C$ is said to be *invertible* if $uu' = u'u = 1$ for some $u' \in C$. In this case, u' is unique, written as $u' =: u^{-1}$ and called the *inverse* of u (in C). The element u is invertible iff L_u is bijective iff R_u is bijective iff U_u is bijective, in which case we have $L_{u^{-1}} = L_u^{-1}$, $R_{u^{-1}} = R_u^{-1}$, $U_{u^{-1}} = U_u^{-1}$. The set of invertible elements in C will be denoted by $C^\times$. It is closed under multiplication; in fact, $(uv)^{-1} = v^{-1}u^{-1}$ for all $u, v \in C^\times$. We also have $C^\times = C^{+\times}$, and the inverses of $u \in C^\times$ in C and C^+ coincide.

1.4 Isotopes. Let $u, v \in C^\times$. Following McCrimmon [9], the algebra given on the k-module C by the multiplication

$$x \cdot_{u,v} y := (xu)(vy) \tag{1.4.1}$$

is written as $C^{(u,v)}$ and is called the *u, v-isotope of C*.

1.5 Theorem. (McCrimmon [9]) *Let C be an alternative algebra and $u, u', v, v' \in C^\times$.*

a) *$C^{(u,v)}$ is an alternative algebra with identity element $1^{(u,v)} = (uv)^{-1}$.*

b) *$C^{(u,v)} = C^{(u',v')}$ iff $u' = us^{-1}$, $v' = sv$ for some $s \in \mathrm{Nuc}(C)^\times$.*

c) *$C^{(u,v)(u',v')} = C^{(u'',v'')}$ for $u'' = u(vu')u, v'' = v(v'u)v$.*

d) *$C^{(u,v)+} = C^{+(uv)}$.* $\square$

2. The structure group

2.0 Setup. Throughout this section, C will be an arbitrary alternative algebra.

2.1 The narrow structure group. Proceeding as in the Jordan case (cf. 1.1), we define the *narrow structure group of* C, denoted by $\mathrm{Str}^{\mathrm{n}}(C)$, as the totality of isomorphisms η from C onto the isotope $C^{(u,v)}$ of C, for some $u, v \in C^\times$ depending on η. It will be seen in due course (but may also be verified directly) that, just as in the Jordan case, $\mathrm{Str}^{\mathrm{n}}(C)$ is a subgroup of $\mathrm{GL}(C)$, the full linear group of C. However, contrary to the Jordan case, u and v cannot be recovered from η; in fact, they are not even uniquely determined (1.5 b)). This simple observation leads to the following concept.

2.2 The structure group. We denote by $\mathrm{Str}(C)$ the set of triples (η, u, v) composed of elements $u, v \in C^\times$ and an isomorphism

$$\eta : C \xrightarrow{\sim} C^{(u,v)}$$

from C onto its u, v-isotope. This amounts to η being a linear bijection and satisfying

$$\eta(xy) = (\eta(x)u)(v\eta(y)) \qquad \text{(by (1.4.1))}. \tag{2.2.1}$$

We also have

$$\eta(1) = 1^{(u,v)} = (uv)^{-1} \qquad \text{(by 1.5 a))}, \tag{2.2.2}$$

and η may be regarded as an isomorphism

$$\eta : C^+ \xrightarrow{\sim} C^{(u,v)+} = C^{+(uv)} \qquad \text{(by 1.5 d))}. \tag{2.2.3}$$

Hence η belongs to structure group of C^+ and, in particular, preserves invertibility. More precisely, by (1.1.6), (1.2.5) and 1.3,

$$\eta(x^{-1}) = (uv)^{-1}\eta(x)^{-1}(uv)^{-1} \qquad (x \in C^\times). \tag{2.2.4}$$

2.3 Theorem. *Let* C *be an alternative algebra. Then* $\mathrm{Str}(C)$ *becomes a group under the multiplication*

$$(\eta, u, v)(\eta', u', v') := (\eta'', u'', v'') \tag{2.3.1}$$

for $(\eta, u, v),\ (\eta', u', v') \in \mathrm{Str}(C)$ *where*

$$\eta'' = \eta\eta', \tag{2.3.2}$$

$$u'' = u(v\eta(u'))u = \eta(1)^{-1}(\eta(u')u) \tag{2.3.3}$$

$$v'' = v(\eta(v')u)v = (v\eta(v'))\eta(1)^{-1}. \tag{2.3.4}$$

$\mathrm{Str}(C)$, *called the* structure group of C, *has the identity element* $(\mathbf{1}_C, 1, 1)$, *and the inverse of* $(\eta, u, v) \in \mathrm{Str}(C)$ *is given by*

$$\begin{aligned}(\eta, u, v)^{-1} &= (\eta^{-1}, \eta^{-1}(v^{-1})^{-1},\ \eta^{-1}(u^{-1})^{-1}) \\ &= (\eta^{-1}, \eta^{-1}(v^{-1}u^{-2}), \eta^{-1}(v^{-2}u^{-1})).\end{aligned} \tag{2.3.5}$$

2.4 Proof of 2.3, Part I. The second equation in (2.3.3) (resp. (2.3.4)) being an immediate consequence of (1.2.4) and (2.2.2), we first show that $\mathrm{Str}(C)$ is closed under the multiplication as described in 2.3. Given $(\eta, u, v), (\eta', u', v') \in \mathrm{Str}(C)$, we define (η'', u'', v'') accordingly. Then $\eta : C \xrightarrow{\sim} C^{(u,v)}$, $\eta' : C \xrightarrow{\sim} C^{(u',v')}$ are isomorphisms. Isotopy being functorial in the obvious sense, η may be viewed as an isomorphism

$$\eta : C^{(u',v')} \xrightarrow{\sim} C^{(u,v)(\eta(u'),\eta(v'))} = C^{(u'',v'')} \qquad \text{(by 1.5 c)),}$$

so $\eta'' = \eta\eta'$ is an isomorphism $C \xrightarrow{\sim} C^{(u'',v'')}$, forcing $(\eta'', u'', v'') \in \mathrm{Str}(C)$, as desired. By a straightforward though somewhat cumbersome computation one can show that $\mathrm{Str}(C)$ does indeed become a group under the operation just defined. We prefer a different approach.

2.5 Albert autotopies. By an *Albert autotopy of C* (cf. Albert [1]) we mean a triple of elements $f, g, h \in \mathrm{GL}(C)$ satisfying

$$f(xy) = g(x)h(y). \tag{2.5.1}$$

Just as in the proof of McCrimmon [9, Theorem 2], we set $y = 1$ to conclude $f(x) = g(x)h(1)$, forcing $h(1) \in C^\times$ and $g(x) = f(x)h(1)^{-1}$. Similarly, $g(1) \in C^\times$ and $h(y) = g(1)^{-1}f(y)$. Thus

$$g = R_{h(1)^{-1}}f, \;\; h = L_{g(1)^{-1}}f, \tag{2.5.2}$$

and (2.5.1) may be rewritten as

$$f(xy) = (f(x)h(1)^{-1})(g(1)^{-1}f(y)). \tag{2.5.3}$$

The totality of Albert autotopies of C will be denoted by $\mathrm{Atp}(C)$; it is obviously a subgroup of $\mathrm{GL}(C)^3$, called the *autotopy group of C*.

2.6 Lemma. *For $\eta \in \mathrm{GL}(C)$ and $u, v \in C^\times$ the following statements are equivalent.*

(i) $(\eta, u, v) \in \mathrm{Str}(C)$.

(ii) *We have*

$$\eta(xy)u = \eta(x)[(uv)(\eta(y)u)]$$

for all $x, y \in C$.

(iii) *We have*

$$v\eta(xy) = [(v\eta(x))(uv)]\eta(y)$$

for all $x, y \in C$.

Proof. By passing to C^{op} if necessary, it suffices to establish the equivalence of (i) and (ii). Assuming (i) we get

$$\begin{aligned}\eta(xy)u &= [(\eta(x)u)(v\eta(y))]u && \text{(by (2.2.1))}\\ &= \eta(x)[(uv)(\eta(y)u)] && \text{(by (1.2.3), (1.2.4))},\end{aligned}$$

and this is (ii). Reversing the argument also gives the opposite implication. $\square$

2.7 Theorem. *The map*

$$\Phi : \mathrm{Atp}(C) \xrightarrow{\sim} \mathrm{Str}(C)$$

defined by

$$\Phi((f,g,h)) := (f, h(1)^{-1}, g(1)^{-1}) \text{ for } (f,g,h) \in \mathrm{Atp}(C)$$

is a group isomorphism satisfying

$$\Phi^{-1}((\eta,u,v)) = (\eta, R_u\eta, L_v\eta) \text{ for } (\eta,u,v) \in \mathrm{Str}(C).$$

2.8 Proof of 2.7 and of 2.3, Part II. We proceed in three steps.
1^0. By (2.5.3), (2.2.1), Φ is well defined. Conversely, (2.2.1) yields a map $\Psi : \mathrm{Str}(C) \longrightarrow \mathrm{Atp}(C)$ given by

$$\Psi((\eta,u,v)) = (\eta, R_u\eta, L_v\eta) \text{ for } (\eta,u,v) \in \mathrm{Str}(C).$$

Using (2.5.2) it is easy to check that $\Psi \circ \Phi$ is the identity on $\mathrm{Atp}(C)$. Conversely, let $(\eta,u,v) \in \mathrm{Str}(C)$. Then

$$\begin{aligned}\Phi \circ \Psi((\eta,u,v)) &= \Phi((\eta, R_u\eta, L_v\eta))\\ &= (\eta, [L_v\eta(1)]^{-1},\ [R_u\eta(1)]^{-1}).\end{aligned}$$

But $L_v\eta(1) = v(uv)^{-1}$ (by (2.2.2)) $= u^{-1}$, $R_u\eta(1) = (uv)^{-1}u = v^{-1}$, so $\Phi \circ \Psi$ is the identity on $\mathrm{Str}(C)$. Therefore Φ is bijective with inverse Ψ.
2^0. Let (η,u,v), $(\eta',u',v') \in \mathrm{Str}(C)$. Defining η'', u'', v'' as in 2.3,

$$\begin{aligned}\Psi((\eta,u,v))\Psi((\eta',u',v')) &= (\eta, R_u\eta, L_v\eta)(\eta', R_{u'}\eta', L_{v'}\eta')\\ &= (\eta\eta', R_u\eta R_{u'}\eta', L_v\eta L_{v'}\eta').\end{aligned}$$

Given $x \in C$, we compute

$$\begin{aligned} R_u \eta R_{u'} \eta'(x) &= \eta(\eta'(x)u')u \\ &= \eta\eta'(x)[(uv)(\eta(u')u)] && \text{(by 2.6)} \\ &= \eta''(x)u'', \end{aligned}$$

$$\begin{aligned} L_v \eta L_{v'} \eta'(x) &= v\eta(v'\eta'(x)) \\ &= [(v\eta(v'))(uv)]\eta\eta'(x) && \text{(by 2.6)} \\ &= v''\eta''(x). \end{aligned}$$

Hence

$$\begin{aligned} \Psi((\eta, u, v))\Psi((\eta', u', v')) &= (\eta'', R_{u''}\eta'', L_{v''}\eta'') \\ &= \Psi((\eta'', u'', v'')), \end{aligned}$$

which shows that $\mathrm{Str}(C)$ is a group and Φ is an isomorphism. This completes the proof of 2.7.
3^0. Since

$$1_{\mathrm{Str}(C)} = \Phi(1_{\mathrm{Atp}(C)}) = \Phi((\mathbf{1}_C, \mathbf{1}_C, \mathbf{1}_C)) = (\mathbf{1}_C, 1, 1),$$

it remains to prove (2.3.5). To do so, we compute

$$\begin{aligned} (\eta, u, v)^{-1} &= \Phi(\Psi((\eta, u, v))^{-1}) && \text{(by } 1^0, 2^0) \\ &= \Phi((\eta, R_u\eta, L_v\eta)^{-1}) \\ &= \Phi((\eta^{-1}, \eta^{-1}R_{u^{-1}}, \eta^{-1}L_{v^{-1}})) \\ &= (\eta^{-1}, \eta^{-1}(v^{-1})^{-1}, \eta^{-1}(u^{-1})^{-1}), \end{aligned}$$

giving the first equation of (2.3.5). As to the second, the relations

$$\begin{aligned} \eta(\eta^{-1}(v^{-1})\eta^{-1}(v^{-1}u^{-2})) &= (v^{-1}u)(v(v^{-1}u^{-2})) && \text{(by (2.2.1))} \\ &= v^{-1}u^{-1} = (uv)^{-1} = \eta(1) && \text{(by (2.2.2))}, \\ \eta(\eta^{-1}(u^{-1})\eta^{-1}(v^{-2}u^{-1})) &= (u^{-1}u)(v(v^{-2}u^{-1})) = \eta(1) \end{aligned}$$

imply $\eta^{-1}(v^{-1})^{-1} = \eta^{-1}(v^{-1}u^{-2}), \eta^{-1}(u^{-1})^{-1} = \eta^{-1}(v^{-2}u^{-1})$, and the proof is complete. $\square$

2.9 The structure group as a group extension. By (2.1) and (2.3.2), the projection onto the first factor gives an epimorphism

(2.9.1) $$\pi : \mathrm{Str}(C) \longrightarrow \mathrm{Str^n}(C), (\eta, u, v) \longmapsto \eta.$$

On the other hand, consulting (2.3.3), (2.3.4), we obtain an embedding

$$\iota : \mathrm{Nuc}(C)^\times \longrightarrow \mathrm{Str}(C),\ s \longmapsto (\mathbf{1}_C, s^{-1}, s), \tag{2.9.2}$$

and 1.5 b) implies that the sequence

$$1 \longrightarrow \mathrm{Nuc}(C)^\times \underset{\iota}{\longrightarrow} \mathrm{Str}(C) \underset{\pi}{\longrightarrow} \mathrm{Str}^{\mathrm{n}}(C) \longrightarrow 1 \tag{2.9.3}$$

is exact. *Thus, for any alternative algebra, the structure group is an extension* (in the group-theoretical sense) *of the narrow structure group by the unit group of the nucleus.*

2.10 Associative algebras. Let C be associative. Then $\mathrm{Nuc}(C) = C$, *and we claim that the short exact sequence* (2.9.3), *i.e.*,

$$1 \longrightarrow C^\times \underset{\iota}{\longrightarrow} \mathrm{Str}(C) \underset{\pi}{\longrightarrow} \mathrm{Str}^{\mathrm{n}}(C) \longrightarrow 1 \tag{2.10.1}$$

splits. In fact, thanks to associativity, a splitting homomorphism is given by

$$\rho : \mathrm{Str}^{\mathrm{n}}(C) \longrightarrow \mathrm{Str}(C),\ \eta \longmapsto (\eta, \eta(1)^{-1}, 1).$$

Hence $\mathrm{Str}(C)$ identifies with the semi-direct product $C^\times \rtimes \mathrm{Str}^{\mathrm{n}}(C)$, where $\mathrm{Str}^{\mathrm{n}}(C)$ acts on $C^\times$ (actually, on C) by automorphisms via

$$\mathrm{Str}^{\mathrm{n}}(C) \times C^\times \longrightarrow C^\times,\ (\eta, s) \longmapsto \eta(s)\eta(1)^{-1}.$$

Incidentally, the narrow structure group is a semi-direct product as well: $\mathrm{Str}^{\mathrm{n}}(C) = C^\times \rtimes \mathrm{Aut}(C)$.

2.11 The connection with the opposite algebra. It follows immediately from 2.3 that the assignment $(\eta, u, v) \longmapsto (\eta, v, u)$ gives an isomorphism from $\mathrm{Str}(C)$ onto $\mathrm{Str}(C^{\mathrm{op}})$ which is compatible with the projections π of (2.9.3). Hence the narrow structure groups of C and C^{op} are the same.

2.12 The connection with Jordan algebras. By 2.2, $\mathrm{Str}^{\mathrm{n}}(C)$ is a subgroup of $\mathrm{Str}(C^+)$. However, the two groups are in general distinct. In fact, as we shall see in 3.10 below, there may even be automorphisms of C^+ which do not belong to the narrow structure group of C.

2.13 Extended left and right multiplications. Given $u \in C^\times$, we put

$$\widetilde{L}_u := (L_u, u, u^{-2}),\ \widetilde{R}_u := (R_u, u^{-2}, u)$$

and call these the *extended left, right multiplication by* u, respectively.

2.14 Proposition. *For alle $u, v \in C^\times$ we have*

a) $\widetilde{L}_u \in \mathrm{Str}(C)$ *and* $\widetilde{L}_{uvu} = \widetilde{L}_u \widetilde{L}_v \widetilde{L}_u, (\widetilde{L}_u)^{-1} = \widetilde{L}_{u^{-1}}$.

b) $\widetilde{R}_u \in \mathrm{Str}(C)$ *and* $\widetilde{R}_{uvu} = \widetilde{R}_u \widetilde{R}_v \widetilde{R}_u, (\widetilde{R}_u)^{-1} = \widetilde{R}_{u^{-1}}$.

c) $\widetilde{L}_u \widetilde{R}_v = (L_u R_v, v^{-2}u, u^{-1}vu^{-1})$.

Proof. a) We verify (2.2.1):

$$\begin{aligned}(L_u(x)u)(u^{-2}L_u(y)) &= (uxu)(u^{-1}y) \\ &= L_u(xy) \qquad \text{(by (1.2.3).}\end{aligned}$$

Hence $\widetilde{L}_u \in \mathrm{Str}(C)$. The equally straightforward verification of the remaining assertions, using $L_{uvu} = L_u L_v L_u$ (by (1.2.3)), is left to the reader.
b) follows from a) by passing to C^{op}.
c) is again left to the reader. □

3. Subgroups of the structure group

3.0 Review. In their work on the classification of Moufang polygons, Tits and Weiss [16] have recently studied a group of transformations defined by octonion division algebras whose definition and elementary properties immediately extend to the more general setting of the present note as follows.

Let C be an alternative algebra. Following Tits-Weiss [16, (36.5)], we denote by X_C the set of all linear bijections $\psi : C \longrightarrow C$ satisfying $u := \psi(1) \in C^\times$ and

$$\psi(xy) = (\psi(x)u^{-1})\psi(y). \tag{3.0.1}$$

Transformations of this kind also arise in Loos [7, §6], describing the connection between alternative pairs and algebras. It is shown in [16, (36.10)] that X_C is a subgroup of $\mathrm{GL}(C)$ which contains all transformations of the form $L_u R_{u^2}$ for $u \in C^\times$ [16, (36.7)]. In the sequel, we wish to identify the group X_C inside the structure group of C.

3.1 One-sided isotopes. The most obvious way to carry out this identification is to look at one-sided isotopes. Indeed, we claim that

$$\mathrm{Str}_l(C) := \{(\eta, u, v) \in \mathrm{Str}(C) \mid v = 1\},$$
$$\mathrm{Str}_r(C) := \{(\eta, u, v) \in \mathrm{Str}(C) \mid u = 1\},$$

are subgroups of $\mathrm{Str}(C)$. With respect to $\mathrm{Str}_l(C)$, for example, this follows easily from 2.3 and the relation

$$\eta(1) = u^{-1} \qquad ((\eta, u, 1) \in \mathrm{Str}_l(C)), \tag{3.1.1}$$

which in turn is an immediate consequence of (2.2.2). Observe too that 2.11 identifies $\mathrm{Str}_r(C)$ with $\mathrm{Str}_l(C^{\mathrm{op}})$. Furthermore, comparing (3.1.1) with (3.0.1), the assignment $\psi \longmapsto (\psi, \psi(1)^{-1}, 1)$ gives an isomorphism from X_C onto the subgroup $\mathrm{Str}_l(C)$ of $\mathrm{Str}(C)$ matching the transformations $L_u R_{u^2}$ of 3.0 with $\widetilde{L}_u \widetilde{R}_{u^2} \in \mathrm{Str}_l(C)$ of 2.14 c).

In the sequel, we will propose another identification of X_C inside the extended structure group which seems to have the advantage of being more symmetric than the previous one.

3.2 Proposition. *Assumptions being as in* 3.0, *the following holds.*

a) *The set*

$$\begin{aligned}\mathrm{Str}_1(C) :&= \{(\eta, u, v) \in \mathrm{Str}(C) \mid v = u^{-1}\}\\ &= \{(\eta, u, v) \in \mathrm{Str}(C) \mid \eta(1) = 1\}\end{aligned}$$

forms a subgroup of $\mathrm{Str}(C)$.

b) *For* $(\eta, u, v) \in \mathrm{Str}_1(C)$, η *is an automorphism of* C^+; *in particular,* η *preserves all powers provided they make sense.*

c) $\mathrm{Int}(u) := \widetilde{L}_u \widetilde{R}_{u^{-1}} = (L_u R_{u^{-1}}, u^3, u^{-3}) \in \mathrm{Str}_1(C)$ *for all* $u \in C^\times$.

Proof. a) The second equation follows from (2.2.2). Hence, by (2.9.3), $\mathrm{Str}_1(C)$ is the pre-image under π of the stabilizer subgroup of 1 in $\mathrm{Str}^{\mathrm{n}}(C)$. This implies a).
b) Given $(\eta, u, v) \in \mathrm{Str}_1(C)$ (so $v = u^{-1}$), we consult (2.2.3) to obtain an isomorphism $\eta : C^+ \xrightarrow{\sim} C^{+(uv)} = C^+$, which proves b).
c) follows by putting $v = u^{-1}$ in 2.14 c). □

3.3 Notation. To simplify notation, we write (η, u) instead of (η, u, u^{-1}) for the elements of $\mathrm{Str}_1(C)$ from now on. Then, by (2.3.3), (2.3.5), the general formulae for the group structure of $\mathrm{Str}(C)$ when restricted to $\mathrm{Str}_1(C)$ reduce to

$$(\eta, u)(\eta', u') = (\eta\eta', \eta(u')u), \tag{3.3.1}$$

$$(\eta, u)^{-1} = (\eta^{-1}, \eta^{-1}(u^{-1})) \tag{3.3.2}$$

for $(\eta, u), (\eta', u') \in \mathrm{Str}_1(C)$. $\square$

3.4 Inner structural transformations. The elements

$$\mathrm{Int}(u) = (L_u R_{u^{-1}}, u^3) \qquad (u \in C^\times) \tag{3.4.1}$$

of $\mathrm{Str}_1(C)$ (cf. 3.2 c)) are called *inner structural transformations.* They may be regarded as the alternative version of inner automorphisms since $C^{(u^3,u^{-3})} = C$ if C is associative. $\square$

3.5 Nuclear and trivial elements. By (2.9.2), $\mathrm{Str}_1(C)$ contains the *nuclear elements* $(\mathbf{1}_C, s)$ for $s \in \mathrm{Nuc}(C)^\times$, so (2.9.3) induces a short exact sequence

$$1 \longrightarrow \mathrm{Nuc}(C)^\times \underset{\iota}{\longrightarrow} \mathrm{Str}_1(C) \underset{\pi}{\longrightarrow} \mathrm{Str}_1^{\mathrm{n}}(C) \longrightarrow 1, \tag{3.5.1}$$

$\mathrm{Str}_1^{\mathrm{n}}(C)$ being the stabilizer group of 1 in $\mathrm{Str}^{\mathrm{n}}(C)$. In addition one checks, using (3.3.1), that the assignment $a \longmapsto (\mathbf{1}_C, a^{-1}1)$ gives a central embedding κ from $k^\times$ to $\mathrm{Str}_1(C)$, allowing us to form $\mathrm{PStr}_1(C) = \mathrm{coker}\ \kappa$. The elements of $\mathrm{im}\ \kappa$ are called *trivial elements of* $\mathrm{Str}_1(C)$. $\square$

3.6 Proposition. *The assignment*

$$(\eta, u, 1) \longmapsto (R_u\eta, u^{-1})$$

defines an isomorphism $F : \mathrm{Str}_l(C) \xrightarrow{\sim} \mathrm{Str}_1(C)$ *whose inverse is given by*

$$F^{-1}((\eta, u)) = (R_u\eta, u^{-1}, 1) \ \ \textit{for} \ \ (\eta, u) \in \mathrm{Str}_1(C).$$

Furthermore, $F(\widetilde{L}_u\widetilde{R}_{u^2}) = \mathrm{Int}(u)$ *for* $u \in C^\times$.

Proof. By 2.6, $(\eta, u, 1) \in \mathrm{GL}(C) \times C^\times \times C^\times$ belongs to $\mathrm{Str}_l(C)$ iff $(R_u\eta, u^{-1})$ belongs to $\mathrm{Str}_1(C)$. Hence F is a well defined map. Bijectivity of F and the precise nature of its inverse being obvious, we show that F is a homomorphism. Given $(\eta, u, 1), (\eta', u', 1) \in \mathrm{Str}_l(C)$, we obtain

$$\begin{aligned} F((\eta, u, 1))F((\eta', u', 1)) &= (R_u\eta, u^{-1})(R_{u'}\eta', u'^{-1}) \\ &= (R_u\eta R_{u'}\eta', [R_u\eta(u'^{-1})]u^{-1}) && \text{(by (3.3.1))} \\ &= (R_u\eta R_{u'}\eta', \eta(u'^{-1})). \end{aligned}$$

Here (2.2.4) yields $\eta(u'^{-1}) = u^{-1}\eta(u')^{-1}u^{-1} = [u\eta(u')u]^{-1}$, and the relation

$$\begin{aligned} R_u\eta R_{u'}\eta'(x) &= \eta(\eta'(x)u')u \\ &= \eta\eta'(x)[u\eta(u'))u] && \text{(by (2.6))} \end{aligned}$$

shows $R_u\eta R_u\eta' = R_{u\eta(u')u}\eta\eta'$, hence

$$\begin{aligned} F((\eta, u, 1))F((\eta', u', 1))) &= (R_{u\eta(u')u}\eta\eta', [u\eta(u')u]^{-1}) \\ &= F((\eta, u, 1)(\eta', u', 1)) && \text{(by 2.3).} \end{aligned}$$

The final assertion follows directly from 2.14 c). □

3.7 Supplement. Combining 3.1 with 3.6, we obtain an identification of X_C with $\mathrm{Str}_1(C)$ under which the maps $L_uR_{u^2}$ $(u \in C^\times)$ of 3.0 correspond to inner structural transformations. □

3.8 Proposition. *Let $(\eta, u) \in \mathrm{Str}_1(C)$ and suppose η fixes u. Then there exists an automorphism φ of C satisfying $\eta^3 = L_uR_{u^{-1}}\varphi$.*

Proof. From $\eta(u) = u$ and (3.3.1) we deduce $(\eta, u)^3 = (\eta^3, u^3)$. Therefore, by 3.2 c), both $L_uR_{u^{-1}}$ and η^3 are isomorphisms from C onto $C^{(u^3,u^{-3})}$, hence differ by an automorphism of C. □

3.9 Proposition. $\mathrm{Str}(C)$ *is generated by extended left multiplications and* $\mathrm{Str}_1(C)$.

Proof. Given $(\eta, u, v) \in \mathrm{Str}(C)$, we conclude $w := \eta(1) \in C^\times$ and $(\eta', u', v') = \widetilde{L}_{w^{-1}}(\eta, u, v) \in \mathrm{Str}_1(C)$. Since $(\eta, u, v) = \widetilde{L}_w(\eta', u', v')$ by 2.14 a), the assertion follows. □

3.10 Proposition. *Assume that C is not commutative and only the scalar multiples of 1 commute with every element of C. Then every algebra anti-automorphism φ of C is an automorphism of C^+ (hence belongs to the structure group of C^+) but does not belong to the narrow structure group of C.*

Proof. The first part is clear. As to the second, assume $\varphi \in \mathrm{Str}^{\mathrm{n}}(C)$. Since φ fixes 1, it even belongs to $\mathrm{Str}^{\mathrm{n}}_1(C)$ and hence, by (3.5.1), is an isomorphism $C \xrightarrow{\sim} C^{(u,u^{-1})}$ for some $u \in C^\times$. But it is also an isomorphism $C \xrightarrow{\sim} C^{\mathrm{op}}$, implying $yx = (xu)(u^{-1}y)$. Putting $y = u$ yields $ux = xu$, hence $u \in k1$ and then $C = C^{\mathrm{op}}$, a contradiction. $\square$

Remark. The hypotheses above are fulfilled for composition algebras of dimension at least 4, with φ the canonical involution (cf. Section 4 below).

3.11 Albert algebras. Elements of the structure group arise naturally in Albert's approach [2] to Albert division algebras as follows. Without striving for maximum generality, we assume from now on that k is a field and consider a cubic Galois extension E/k, with generating Galois automorphism ρ. We denote by C the split octonion algebra over E and write n for its norm, t for its trace, $*$ for its canonical involution, a good reference for octonion algebras being Springer-Veldkamp [15]. We let $H_3(C)$ stand for the split Albert algebra over E, consisting of all 3-by-3 $*$-hermitian matrices with entries in C and scalars (in E) down the diagonal; it is a quadratic Jordan algebra over E in a natural way, cf. McCrimmon [8]. We find it convenient to identify E in $H_3(C)$ by matching $\alpha \in E$ with the diagonal matrix $\mathrm{diag}(\alpha, \rho(\alpha), \rho^2(\alpha)) \in H_3(C)$. Referring to [6], [12] for details, we are interested in pairs (J, φ) having the following properties.

(3.11.1) J is an Albert algebra arising from the first Tits construction.

(3.11.2) $\varphi : E \longrightarrow J$ is an embedding.

Isomorphisms between such pairs are defined in the obvious way.

On the other hand, we consider pairs (σ, u) such that the following conditions hold.

(3.11.3) $\sigma : C \longrightarrow C$ is ρ-semilinear,

(3.11.4) $u \in C,\ n(u) = 1,$ and $\sigma(u) = u$,

(3.11.5) $\sigma(xy)u = \sigma(x)(\sigma(y)u)$,

(3.11.6) $\sigma^3(x) = uxu^{-1}$.

It follows from the work of Albert [2] that every pair (σ, u) satisfying (3.11.3)-(3.11.6) naturally determines a ρ-semilinear automorphism of $H_3(C)$ having order 3, which in turn, by passing to the fixed points, induces a pair (J, φ) satisfying (3.11.1), (3.11.2). Conversely, every such pair up to isomorphism arises in this way.

Now suppose (σ, u) satisfies (3.11.3) - (3.11.6). Regarding C as an alternative algebra over k, we may apply 2.6 with $v = u^{-1}$ to conclude from (3.11.5) that (σ, u) belongs to $\mathrm{Str}_1(C)$. By 3.8, (3.11.3) - (3.11.5) alone imply that σ^3 differs from $L_u R_{u^{-1}}$ by an automorphism φ of C, so (3.11.6) amounts to the additional normalization $\varphi = \mathbf{1}_C$.

3.12 Algebraic groups. Assuming for simplicity that k is an algebraically closed field, we briefly study the preceding concepts in the setting of algebraic groups, a good reference to this topic being Springer [14]. Let C be an alternative algebra of finite dimension. Then $\mathrm{Str}(C)$ is obviously a linear algebraic group, and the projection π of (2.9.1) gives a morphism $\mathrm{Str}(C) \longrightarrow \mathrm{GL}(C)$ of algebraic groups. Therefore the narrow structure group of C, being the image of $\mathrm{Str}(C)$ under this morphism, is a closed subgroup of $\mathrm{GL}(C)$ [14, 2.2.5 (ii)] and hence a linear algebraic group in its own right. We thus arrive at (2.9.3) as a short exact sequence in the category of algebraic groups. One may ask when this sequence splits. It certainly does if C is associative, i.e., if its nucleus is big enough (2.10). On the other hand, if the nucleus is small, and the degree of C as an algebra avoids certain isolated values, we obtain the following result.

3.13 Proposition. *Notations and assumptions being as in* 3.12, *suppose that* $\mathrm{Nuc}(C) = k1$ *and the degree of* C *is different from* 1 *and* 3. *Then there is no morphism* $\rho : \mathrm{Str}^{\mathrm{n}}(C) \longrightarrow \mathrm{Str}(C)$ *of affine algebraic sets satisfying* $\pi \circ \rho = \mathbf{1}_{\mathrm{Str}^{\mathrm{n}}(C)}$. *In particular,* (2.9.3) *does not split in the category of algebraic groups.*

Proof. Assume the contrary, so there is a morphism ρ of affine algebraic sets as above. Given $\eta \in \mathrm{Str}^{\mathrm{n}}(C)$, we therefore obtain $\rho(\eta) = (\eta, u(\eta), v(\eta))$, where $u(\eta), v(\eta) \in C^\times$ have the property that $\eta : C \xrightarrow{\sim} C^{(u(\eta), v(\eta))}$ is an isomorphism. Applying this to $\eta = L_x R_{x^{-1}}$ $(x \in C^\times)$ and comparing $\rho(\eta)$ with $\mathrm{Int}(x)$ by means of 1.5 b), (3.4.1), we find a map $\mu : C^\times \longrightarrow k^\times$ satisfying $v(L_x R_{x^{-1}}) = \mu(x) x^{-3}$. The left-hand side being homogeneous of degree 0, μ is homogeneous of degree 3 and, at the same time, a morphism of affine algebraic sets. Viewed as a k-valued function, μ therefore belongs to the group of units in the coordinate algebra of $C^\times$, which is just the polynomial ring over k in $m = \dim C$ variables localized at n, the generic norm of C. This implies $\mu = a n^r$ for some $a \in k^\times$, $r \in \mathbb{Z}$, and comparing degrees yields $3 = r \deg C$, a contradiction. $\square$

Remark. a) The same argument shows that 3.13 continues to hold when (2.9.3) is replaced by (3.5.1).
b) The hypotheses of 3.13 are fulfilled if C is an octonion algebra. But even in that case I don't know whether (2.9.3) splits in the category of *abstract* groups.

3.14 Degenerate examples. As yet another illustration, we discuss a class of highly degenerate examples; proofs will be the exception rather than the rule. Letting k again be an arbitrary commutative associative ring of scalars and l, m, r be positive integers we consider the polynomial ring $S = k[X_0, \ldots, X_r]$, equipped with its natural grading

$$S = \bigoplus_{d \geq 0} S_d.$$

Following [11, 3.8], the k-module

$$C = \begin{pmatrix} k & S_l \oplus S_m \\ S_{l+m} & k \end{pmatrix}$$

becomes an alternative algebra of degree 2 (cf. McCrimmon [10]) under the multiplication

$$\begin{pmatrix} a & f_l \oplus f'_m \\ f_{l+m} & a' \end{pmatrix} \begin{pmatrix} b & g_l \oplus g'_m \\ g_{l+m} & b' \end{pmatrix} = \begin{pmatrix} ab & h_l \oplus h'_m \\ h_{l+m} & a'b' \end{pmatrix},$$

where $a, a', b, b' \in k$, the $f's, g's$ are homogeneous polynomials in S, with subscripts indicating their respective degrees, and the $h's$ are determined by the formulae

$$h_l = ag_l + b'f_l,\ h'_m = ag'_m + b'f'_m,$$
$$h_{l+m} = bf_{l+m} + a'g_{l+m} + f_l g'_m - f'_m g_l.$$

The norm of C (as an algebra of degree 2) as given by

$$n(\begin{pmatrix} a & f_l \oplus f'_m \\ f_{l+m} & a' \end{pmatrix}) = aa'.$$

The following result derives from a straightforward but somewhat lengthy computation.

3.15 Proposition. *Notations being as in* 3.14,

$$\mathrm{Nuc}(C) = \{\begin{pmatrix} a & 0 \\ f_{l+m} & a \end{pmatrix} \mid a \in k,\ f_{l+m} \in S_{l+m}\},$$

and this is a commutative associative subalgebra of C. *Furthermore,* C *is central:* $\mathrm{Cent}(C) = k1$. □

3.16 Isotopy and isomorphism. Given any alternative algebra C and $u, v \in C^\times$, it is natural to ask whether C and $C^{(u,v)}$ are isomorphic. Aside from an example constructed by McCrimmon [9, p. 259], utilizing certain pathologies in characteristic 3, no alternative algebras are known where this fails to hold. Also, in dealing with this question, one may always assume $v = u^{-1}$ since, setting $w = u^2 v$,

$$L_{uv} : C^{(u,v)} \xrightarrow{\sim} C^{(w,w^{-1})}$$

is easily seen to be an isomorphism. Using this, we can prove:

3.17 Proposition. *Notations being as in* 3.14, *all isotopes of C are isomorphic.*

Proof. Given $u, v \in C^\times$, we must show that C and $C^{(u,v)}$ are isomorphic. By 3.16, we may assume $v = u^{-1}$. Then we put

$$u = \begin{pmatrix} a & f_l \oplus f'_m \\ f_{l+m} & a' \end{pmatrix}$$

as in 3.14, where $a, a' \in k^\times$. Multiplying u from the right by an invertible element of the nucleus (cf. 3.15), which we are allowed to do by 1.5 b), we may assume $f_{l+m} = 0,\ a' = 1$. Then one checks, again by a straightforward but lengthy computation, that the assignment

$$\begin{pmatrix} b & g_l \oplus g'_m \\ g_{l+m} & b' \end{pmatrix} \longmapsto \begin{pmatrix} b & g_l \oplus g'_m \\ a^{-1}[g_{l+m} + f_l g'_m - f'_m g_l] & b' \end{pmatrix}$$

gives an isomorphism from C onto $C^{(u,u^{-1})}$, as desired. $\square$

Remark. Even though C as in 3.14 is a highly degenerate alternative algebra (for example, if k is a field, the radical of C has codimension 2 in C), its structure group by 3.17 is still big enough to ensure that all isotopes are isomorphic.

4. Composition algebras

4.0 Standard properties. Throughout this section k is assumed to be a field. Let C be a composition algebra with norm n, trace t and canonical involution $*$, the bilinearization of the norm being written as $n(x, y)$. Again a good reference is Springer-Veldkamp [15]. In addition we recall that C^+, the

Jordan algebra corresponding to C (cf. 1.2), agrees with the Jordan algebra corresponding to the quadratic form n with base point 1, so

$$xyx = U_x y = n(x, y^*)x - n(x)y^*. \tag{4.0.1}$$

We also note

$$t(xy) = n(x, y^*). \tag{4.0.2}$$

Finally, $x \in C$ is invertible if and only if $n(x) \neq 0$, in which case

$$x^{-1} = n(x)^{-1}x^*. \tag{4.0.3}$$

Writing S_v for the reflection (resp. transvection in characteristic two) effected by $v \in C^\times$, (4.0.1) implies

$$U_v = n(v)S_v S_1 \qquad (v \in C^\times). \tag{4.0.4}$$

Our ultimate goal will be to provide a convenient set of generators for the group $\mathrm{Str}_1(C)$ when C is an octonion algebra; by 3.1, 3.6, 3.9, this will automatically yield generators both for the structure group and for the group of Tits and Weiss. At a critical stage, our approach will rely heavily on the principle of triality (cf. Springer-Veldkamp [15]).

4.1 Proposition. *The assignment* $(\eta, u) \longmapsto n(u)$ *determines a homomorphism* $\mathrm{Str}_1(C) \longrightarrow k^\times$. *In particular,*

$$\mathrm{Str}_0(C) := \{(\eta, u) \in \mathrm{Str}_1(C) \mid n(u) = 1\}$$

is a normal subgroup of $\mathrm{Str}_1(C)$. *Every element* (η, u) *of* $\mathrm{Str}_1(C)$ *allows a decomposition*

$$(\eta, u) = (\mathbf{1}_C, n(u)^{-1}1)\mathrm{Int}(u)(\eta_0, u_0) \tag{4.1.1}$$

where

$$(\eta_0, u_0) = (R_u L_{u^{-1}}\eta, u^* u^{-1}) \in \mathrm{Str}_0(C).$$

In particular, $\mathrm{Str}_1(C)$ *is generated by trivial elements, inner structural transformations and* $\mathrm{Str}_0(C)$.

Proof. For all $(\eta, u) \in \mathrm{Str}_1(C)$, η is an automorphism of C^+ (3.2 b)) and so belongs to $O(C, n)$, the orthogonal group of the quadratic space (C, n)

(Jacobson-McCrimmon [5, Theorem 6]). Hence the first part of 4.1 follows from (3.3.1). As to (4.1.1), we compute

$$\begin{aligned}
&\mathrm{Int}(u)^{-1}(\mathbf{1}_C, n(u)^{-1}1)^{-1}(\eta, u) \\
&\qquad = (R_u L_{u^{-1}}, u^{-3})(\mathbf{1}_C, n(u)1)(\eta, u) && \text{(by (3.3.2))} \\
&\qquad = (R_u L_{u^{-1}}, u^{-3})(\eta, n(u)u) && \text{(by (3.3.1),} \\
&\qquad = (R_u L_{u^{-1}}\eta, n(u)uu^{-3}) = (\eta_0, u_0).
\end{aligned}$$

Hence (η_0, u_0) belongs to $\mathrm{Str}_1(C)$, even to $\mathrm{Str}_0(C)$ since $n(u_0) = 1$, and (4.1.1) holds. ☐

4.2 Proposition. *Let $r \in \mathbb{N}$ and suppose $(v_1, \ldots, v_r)$ is a* $*$-close *sequence of invertible elements in C, so*

$$u := v_1(v_2(\ldots(v_{r-1}v_r)\ldots)) = \varepsilon v_1^*(v_2^*(\ldots(v_{r-1}^* v_r^*)\ldots)) \tag{4.2.1}$$

for some $\varepsilon \in \{\pm 1\}$. Then

$$\eta := \varepsilon L_{u^{*-1}} \prod_{j=1}^{r} R_{v_j} \tag{4.2.2}$$

satisfies

$$\eta = \varepsilon n(u)^{-1} \prod_{j=1}^{r} U_{v_j} = R_{u^{-1}} \prod_{j=1}^{r} L_{v_j}, \tag{4.2.3}$$

and $(\eta, u) \in \mathrm{Str}_1(C)$.

Proof. We thave $u^{*-1} = n(u)^{-1}u$ by (4.0.3). Hence repeated application of 1.2.4 gives

$$\begin{aligned}
\eta(x) &= \varepsilon n(u)^{-1}[v_1(v_2(\ldots(v_{r-1}v_r)\ldots))][((\ldots((xv_r)v_{r-1})\ldots)v_2)v_1] \\
&= \varepsilon n(u)^{-1} v_1(v_2(\ldots(v_{r-1}(v_r x v_r)v_{r-1})\ldots)v_2)v_1 \\
&= \varepsilon n(u)^{-1} \prod_{j=1}^{r} U_{v_j} x.
\end{aligned}$$

Thus the first equation of (4.2.3) holds. Similarly,

$$\begin{aligned}
R_{u^{-1}} \prod_{j=1}^{r} L_{v_j} x = n(u)^{-1} R_{u^*} \prod_{j=1}^{r} L_{v_j} x \\
&= \varepsilon n(u)^{-1} [v_1(v_2(\dots(v_{r-1}(v_r x))\dots))][((\dots(v_r v_{r-1})\dots)v_2)v_1] \\
&= \varepsilon n(u)^{-1} v_1(v_2(\dots(v_{r-1}(v_r x r_r)v_{r-1})\dots)v_2)v_1 \\
&= \varepsilon n(u)^{-1} \prod_{j=1}^{r} U_{v_j} x,
\end{aligned}$$

yielding the second equation of (4.2.3) as well. This implies

$$\begin{aligned}
\eta(xy) &= \varepsilon n(u)^{-1} \prod_{j=1}^{r} U_{v_j}(xy) \\
&= \varepsilon n(u)^{-1} v_1(v_2(\dots(v_{r-1}(v_r(xy)v_r)v_{r-1})\dots)v_2)v_1 \\
&= \varepsilon n(u)^{-1} [v_1(v_2(\dots(v_{r-1}(v_r x))\dots))][((\dots((y v_r)v_{r-1})\dots)v_2)v_1] \\
&= \left(\prod_{j=1}^{r} L_{v_j} x\right)\left(\varepsilon n(u)^{-1} \prod_{j=1}^{r} R_{v_j} y\right) \\
&= (\eta(x)u)(u^{-1}\eta(y))
\end{aligned}$$

by (4.2.2), (4.2.3). Hence $(\eta, u) = (\eta, u, u^{-1}) \in \mathrm{Str}_1(C)$, as claimed. $\square$

4.3 Theorem. *Suppose C is an octonion algebra over k, with norm n and canonical involution $*$. Then $\mathrm{Str}_0(C)$ consists of all elements (η, u) where $u \in C$ satisfies*

(4.3.1) $$n(u) = 1$$

as well as

(4.3.2) $$u = v_1(v_2(\dots(v_{r-1}v_r)\dots)) = v_1^*(v_2^*(\dots(v_{r-1}^* v_r^*)\dots))$$

for some $r \in \mathbb{N}$, $v_1, \dots, v_r \in C^\times$, and

(4.3.3) $$\eta = L_u \prod_{j=1}^{r} R_{v_j}.$$

Proof. All transformations of this form belong to $\mathrm{Str}_0(C)$, by 4.2 and (4.0.3). Conversely, given $(\eta, u) \in \mathrm{Str}_0(C)$, we conclude $n(u) = 1$ and

$$\eta_1(xy) = \eta_2(x)\eta_3(y),$$

where $\eta_1 = \eta,\ \eta_2 = R_u\eta,\ \eta_3 = L_{u^{-1}}\eta$ all belong to $O(C, n)$ (3.2 b) and [5, Theorem 6]). The principle of triality [15, Theorem 3.3.1], combined with its supplement [15, Proposition 3.3.2], therefore implies that η belongs to the reduced orthogonal group of n and hence allows a representation

$$\eta = \prod_{j=1}^{r} S_{w_j},\ \prod_{j=1}^{r} n(w_j) = 1$$

with an even number of invertible elements $w_1, \ldots, w_r \in C$. By direct computation, which goes back to Jacobson [4, p. 75 (resp. 408)], (4.0.1) now yields $v_1, \ldots, v_r \in C^\times$ satisfying

$$\eta = \prod_{j=1}^{r} U_{v_j},\ \prod_{j=1}^{r} n(v_j) = 1. \tag{4.3.4}$$

Consulting (1.2.4) and the triality principle again [15, Theorem 3.3.1, (iii)], we conclude

$$R_u\eta = a\prod_{j=1}^{r} L_{v_j},\ L_{u^{-1}}\eta = a^{-1}\prod_{j=1}^{r} R_{v_j}$$

for some $a \in k^\times$. Furthermore, by 3.2 a), η fixes 1, which implies

$$u = R_u\eta(1) = av_1(v_2(\ldots(v_{r-1}v_r)\ldots)),$$
$$u^* = L_{u^{-1}}\eta(1) = a^{-1}((\ldots(v_rv_{r-1})\ldots)v_2)v_1.$$

Taking norms and observing (4.3.4), we conclude $a = \pm 1$, so after replacing v_1 by av_1 if necessary, we arrive at the desired conclusions. □

4.4 Corollary. *Assumptions being as in* 4.3, $\mathrm{Str}_1(C)$ *is generated by trivial elements, inner structural transformations and elements of the form* (η, u)*, where* u *and* η *satisfy* (4.3.1) - (4.3.3).

Proof. Immediate from 4.1, 4.3. □

Remark. The sticky point with the set of generators for $\mathrm{Str}_1(C)$ presented in 4.4 is that it seems difficult to obtain a handy description of elements $v_1, \ldots, v_r \in C^\times$ ($r \in \mathbb{N}$) satisfying the restrictions (4.3.1), (4.3.2). However, if we relax these restrictions by allowing a sign factor $\varepsilon \in \{\pm 1\}$ in (4.3.2) and by dropping (4.3.1) altogether, which continues to produce elements of $\mathrm{Str}_1(C)$ (4.2), such a description is indeed possible. In fact, as we shall see below, elements $v_1, \ldots, v_r \in C^\times$ ($r \in \mathbb{N}$) satisfying (4.2.1) exist in abundance. The key to this observation will be the following version of Hilbert's Theorem 90 for composition algebras. □

4.5 Proposition. *Notations being as in* 4.0, *let* $z \in C \setminus \{\pm 1\}$. *Then the following statements are equivalent.*

(i) *There are elements* $v \in C^\times$, $\varepsilon \in \{\pm 1\}$ *such that* $z = \varepsilon v v^{*-1}$.

(ii) $n(z) = 1$, *and if* k *has characteristic two, then* $t(z) \neq 0$.

In this case, ε *remaining fixed,* v *in* (i) *is unique up to a nonzero scalar factor.*

Proof. (i) $\Longrightarrow$ (ii). We clearly have $n(z) = 1$ and $z \notin k1$, so z generates a two-dimensional subalgebra $k[z] \subset C$. Also, $z \in k[v]$, and comparing dimensions yields $k[z] = k[v]$, hence $v \in k[z]$. Assuming char $k = 2$ and $t(z) = 0$ would force t to vanish identically on $k[z]$ since $t(1) = 2 = 0$. In particular, $t(v) = 0$, hence $v = v^*$ and $z = 1$, a contradiction.
(ii) $\Longrightarrow$ (i). Again $k[z]$ is a two-dimensional subalgebra of C. If $k[z]/k$ is étale, Hilbert's Theorem 90 in its classical form implies (i) with $\varepsilon = 1$. If $k[z]/k$ is not étale, it is either the algebra of dual numbers or a purely inseparable field extension of characteristic two, the latter possibility, as well as the former for char $k = 2$, being excluded by the assumption $t(z) \neq 0$. Hence char $k \neq 2$ and $z = \varepsilon 1 + w$ for some $\varepsilon \in \{\pm 1\}$ and $w \in C$ satisfying $w^2 = 0 \neq w$. Then (i) holds with $v = \varepsilon 1 + \frac{1}{2} w$.
To establish the final statement of the proposition, we fix ε and note by one of the preceding arguments that *any* v satisfying (i) belongs to $k[z]$. It therefore suffices to show that the homomorphism

$$k[z]^\times \longrightarrow k[z]^\times,\ x \longmapsto x x^{*-1}$$

has kernel $k^\times 1$. This in turn amounts to $x \in k1$ for $x = x^* \in k[z]$, which is clear in all characteristics since $t(z) \neq 0$ for char $k = 2$. □

4.6 $*$-close sequences. Formalizing earlier notation, a finite sequence $(v_1, \ldots, v_r)$ of invertible elements in C will be called $*$-*close* if it satisfies (4.2.1) for some $\varepsilon \in \{\pm 1\}$. We wish to construct $*$-close sequences in large quantities. Rather than doing so directly, we first aim at exhibiting situations where both sides of (4.2.1) are sufficiently far apart. The meaning of this will be made precise in the following definition.

4.7 $*$-remote sequences. A finite sequence $(v_1, \ldots, v_r)$ of invertible elements in C is said to be $*$-*remote* if either char $k \neq 2$ and

$$v_1(\ldots(v_{r-1}v_r)\ldots) \neq \pm v_1^*(\ldots(v_{r-1}^*v_r^*)\ldots) \tag{4.7.1}$$

or char $k = 2$ and

$$(4.7.2) \qquad v_1(\ldots(v_{r-1}v_r)\ldots) + v_1^*(\ldots(v_{r-1}^*v_r^*)\ldots) \in C^\times.$$

Note that, for C a division algebra of characteristic two, (4.7.2) is the same as (4.7.1). We wish to prove that, aside from minor exceptions, any finite sequence of invertible elements may be extended to a $*$-remote one by adding just a single term. This follows easily if the base field is infinite, thanks to Zariski density, but requires more care in general. In what follows, the algebra of 2×2-matrices will be denoted by $\mathrm{Mat}_2(k)$. We begin with an easy lemma, whose proof will be omitted.

4.8 Lemma. *Precisely one of the following holds.*

a) *There exists an invertible element $v \in C$ satisfying $v^* \neq \pm v$.*

b) $\dim C = 1$.

c) *$k \cong \mathbb{F}_2$ or $k \cong \mathbb{F}_3$, and $C \cong k \times k$.* □

4.9 Lemma. *If* char $k = 2$ *and $x, y \in C$ satisfy $n(x) = n(y)$, then*

$$n(vx + v^*y) = t(v)n(v, yx^*) - n(v)n(x, y)$$

for all $v \in C$.

Proof. Expanding the left-hand side gives

$$\begin{aligned} n(vx + v^*y) &= n(v)n(x) + n(vx, v^*y) + n(v)n(y) \\ &= t\big((vx)(y^*v)\big) && \text{(by (4.0.2))} \\ &= t\big(v(xy^*)v\big) && \text{(by (1.2.4))} \\ &= t\big(n(v, yx^*)v - n(v)yx^*\big) && \text{(by (4.0.1))} \\ &= t(v)n(v, yx^*) - n(v)n(x, y). \end{aligned}$$ □

4.10 Proposition. *Let $(v_2, \ldots, v_r)$ be a finite sequence of invertible elements in C and put*

$$w := [v_2(\ldots(v_{r-1}v_r)\ldots)][(\ldots(v_rv_{r-1})\ldots)v_2].$$

Then precisely one of the following holds.

a) $(v_2, \dots, v_r)$ *extends to a $*$-remote sequence* $(v_1, v_2, \dots, v_r)$ *in* C, *for some* $v_1 \in C^\times$.

b) $\dim C = 1$.

c) $k \cong \mathbb{F}_2$ *or* $k \cong \mathbb{F}_3$, *and* $C \cong k \times k$.

d) $k \cong \mathbb{F}_2$ *and* $C \cong \mathrm{Mat}_2(k)$, $w \notin k1$, $t(w) = 0$.

Proof. We put

$$x = v_2(\dots(v_{r-1}v_r)\dots), \; y = v_2^*(\dots(v_{r-1}^*v_r^*)\dots)$$

and consult 4.9, (4.0.2) to conclude

$$n(x+y) = n(x,y) = t(w) \qquad (\mathrm{char}\, k = 2). \tag{4.10.1}$$

Returning to the general setup, we first consider the case that $(v_2, \dots, v_r)$ is $*$-remote. Then a) holds (put $v_1 = 1$) but b) doesn't. Neither does c) for $k \cong \mathbb{F}_3$ by 4.8 since $x^* \neq \pm x$. Also, assuming d) or c) for $k \cong \mathbb{F}_2$ gives $t(w) \neq 0$ by (4.10.1) and $*$-remoteness, a contradiction. We may therefore assume for the rest of the proof that $(v_2, \dots, v_r)$ is not $*$-remote. Then we consider the following cases.

Case 1. char $k \neq 2$.
Then $v_1 \in C^\times$ makes $(v_1, v_2, \dots, v_r)$ $*$-remote if and only if $v_1^* \neq \pm v_1$. Hence the assertion follows from 4.8.

Case 2. char $k = 2$.
Since $(v_2, \dots, v_r)$ is not $*$-remote, (4.10.1) gives

$$n(x,y) = t(w) = 0. \tag{4.10.2}$$

From this and 4.9 we conclude

$$\text{a)} \iff \exists\, v \in C^\times : t(v) \neq 0 \neq t(vw). \tag{4.10.3}$$

The existence of such an element v will now be discussed in the following subcases.

Case 2.1. $w \in k1$.
By (4.10.3), a) holds iff $t(v) \neq 0$ for some $v \in C^\times$ iff c) doesn't hold.

Case 2.2. $w \notin k1$.
If C is an octonion algebra, we choose a quaternion subalgebra $B \subset C$

containing w [15, Proposition 1.6.4] and find an element $u \in B$ satisfying $t(u) \neq 0 \neq t(uw)$. If u is invertible, we are done, by (4.10.3). If not, we can find an invertible element $z \in B^{\perp}$, the orthogonal complement of B relative to n, and $v = u + z \in C^{\times}$ satisfies $t(v) \neq 0 \neq t(vw)$, as desired. If C is not an octonion algebra, it must have dimension 4 since, otherwise, by (4.10.2), its trace form would be identically zero. As before, we obtain $t(v) \neq 0 \neq t(vw)$ for some $v \in C$, allowing us to assume that $C = \mathrm{Mat}_2(k)$ is split. Then the minimum polynomial of w has the form $\lambda^2 + n(w)$, so if $n(w) \in k^{\times}$ is not a square, the subalgebra of C generated by w is an inseparable quadratic field extension, forcing k to be infinite. Hence Zariski density allows us to assume $v \in C^{\times}$, and we are done again. Finally, if $n(w)$ happens to be a square, we may assume that w has the form

$$w = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

for some $a \in k^{\times}$. This is easily seen to imply a) unless $k \cong \mathbb{F}_2$, in which case we have d). □

Having moved things sufficiently far apart, it is now easy to bring them close together:

4.11 Theorem. *Let $(v_2, \ldots, v_r)$ be a $*$-remote sequence of invertible elements in C. Then there exist $v_1 \in C^{\times}$ and $\varepsilon \in \{\pm 1\}$ satisfying*

$$v_1\bigl(v_2(\ldots(v_{r-1}v_r)\ldots)\bigr) = \varepsilon v_1^*\bigl(v_2^*(\ldots(v_{r-1}^*v_r^*)\ldots)\bigr). \tag{4.11.1}$$

Furthermore, ε remaining fixed, v_1 is unique up to a nonzero scalar factor.

Proof. We put

$$x := v_2\bigl(\ldots(v_{r-1}v_r)\ldots\bigr),\ y := v_2^*\bigl(\ldots(v_{r-1}^*v_r^*)\ldots\bigr),$$

$z := yx^{-1}$ and, given $v_1 \in C^{\times}$, $\varepsilon \in \{\pm 1\}$, obtain the following chain of equivalent statements.

$$\begin{aligned}
(4.11.1) &\iff v_1 x = \varepsilon v_1^* y \\
&\iff y = \varepsilon v_1^{*-1}(v_1 x) \\
&\iff y = \varepsilon n(v_1)^{-1} v_1^2 x && \text{(by (4.0.3), (1.2.2))} \\
&\iff z = \varepsilon v_1 v_1^{*-1}.
\end{aligned}$$

By 4.5 it therefore suffices to show that $z \neq \pm 1$ satisfies (ii) of that proposition. The condition $n(z) = 1$ being obvious, $*$-remoteness implies $z \neq \pm 1$

for char $k \neq 2$, whereas, for char $k = 2$, it combines with 4.9 and (4.0.2) to yield $t(z) = n(x)^{-1}n(x,y) \neq 0$, hence $z \neq \pm 1$ again. $\square$

Remark. Given a $*$-remote sequence $(v_2, \ldots, v_r)$ of invertible elements in C and $\varepsilon \in \{\pm 1\}$ such that $v_1 \in C^\times$ satisfying (4.11.1) exists (otherwise pass to $-\varepsilon$), the element of $\mathrm{Str}_1(C)$ (cf. 3.4) obtained from the $*$-close sequence $(v_1, v_2, \ldots, v_r)$ via (4.2.2) does not depend on the choice of v_1. $\square$

4.12 Corollary. *Let $v_4, \ldots, v_r$ be invertible elements of C, and suppose C is an octonion algebra. Then there are $v_1, v_2, v_3 \in C^\times$ satisfying*

$$v_1(v_2(v_3(v_4(\ldots(v_{r-1}v_r)\ldots)))) = v_1^*(v_2^*(v_3^*(v_4^*(\ldots(v_{r-1}^*v_r^*)\ldots)))).$$

Proof. Immediate from 4.10, 4.11. $\square$

Remark. Thanks to 4.9, sequences $(v_1, \ldots, v_r)$ $(r \geq 4)$ of invertible elements in C satisfying (4.3.2) are easily constructed. However, if the subsequence $(v_2, \ldots, v_r)$ obtained in this construction happens to be $*$-remote, we lose control over the validity of (4.3.1) since, by 4.11, $n(u)$ is unique up to a nonzero square factor.

4.13 Conclusion: Octonion algebras. Notations being as in 4.0, we briefly describe the structure group, as well as the various groups related to it, in case C is an octonion algebra. To do so, we systematically identify $\mathrm{Str}(C) = \mathrm{Atp}(C)$ via 2.7. Then the principle of triality, [15, 3.2.1] yields an identification of $\mathrm{Str}^{\mathrm{n}}(C)$ with $\mathrm{SGO}(C,n)$, the group of special similarities of the quadratic space (C,n). Hence $\mathrm{Str}^{\mathrm{n}}_1(C)$ becomes the stabilizer $\mathrm{SGO}_1(C,n)$ of 1 in $\mathrm{SGO}(C,n)$, and by (2.9.3), (3.5.1), the groups $\mathrm{Str}(C), \mathrm{Str}_1(C)$ fit into exact sequences

$$1 \longrightarrow k^\times \longrightarrow \mathrm{Str}(C) \longrightarrow \mathrm{SGO}(C,n) \longrightarrow 1,$$
$$1 \longrightarrow k^\times \longrightarrow \mathrm{Str}_1(C) \longrightarrow \mathrm{SGO}_1(C,n) \longrightarrow 1,$$

respectively. Finally, by [4, §§ 2,6] and [15, 3.6], there are three inequivalent representations χ, ρ_1, ρ_2 of $\mathrm{Spin}(C,n)$ on C, χ being derived from the vector representation of the Clifford group, which are irreducible and canonically determine an isomorphism

$$\mathrm{Spin}(C,n) \xrightarrow{\sim} \mathrm{Atp}(C) \cap \mathrm{O}(C,n)^3.$$

It now follows easily that $\mathrm{Str}_0(C)$ is a subgroup of $\mathrm{Atp}(C) \cap \mathrm{O}(C,n)^3$ which, under this isomorphism, corresponds to the subgroup

$$\mathrm{Spin}_0(C,n) = \{x \in \mathrm{Spin}(C,n) | \chi(x)(1) = 1\}$$

of $\mathrm{Spin}(C)$. $\square$

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